Grandes déviations pour des équations de Schrödinger non linéaires stochastiques et applications

Eric Gautier

N° d'Ordre : 3264

THÈSE

Présentée

DEVANT L'UNIVERSITÉ DE RENNES I

pour obtenir

le grade de DOCTEUR DE L'UNIVERSITÉ DE RENNES I

Mention Mathématiques et Applications

par

Eric GAUTIER

Institut de Recherche Mathématique de Rennes
Ecole Normale Supérieure de Cachan, Antenne de Bretagne
&
Centre de Recherche en Economie et Statistique - INSEE

École Doctorale MATISSE

U.F.R. de Mathématiques

TITRE DE LA THÈSE :

Grandes déviations pour des équations de Schrödinger non linéaires stochastiques et applications

Soutenue le 09 décembre 2005 devant la Commission d'Examen

COMPOSITION DU JURY :

Mme	Anne DE BOUARD	Examinatrice
M	François COQUET	Examinateur, membre invité
M	Arnaud DEBUSSCHE	Directeur de thèse
M	Josselin GARNIER	Rapporteur
M	Ying HU	Examinateur
Mme	Sylvie MÉLÉARD	Examinatrice
Mme	Marta SANZ-SOLÉ	Rapportrice

Remerciements

Dans un premier temps, je souhaiterais exprimer ma profonde gratitude à mon directeur de thèse Arnaud Debussche. Merci de m'avoir fait confiance et de m'avoir fourni un sujet que j'ai trouvé tout à fait passionnant, à la frontière entre les probabilités, les équations aux dérivées partielles et la physique. J'ai le sentiment d'avoir appris beaucoup de belles mathématiques. Je le remercie aussi pour les précieuses qualités scientifiques et pédagogiques de son encadrement et pour sa grande disponibilité.

Je remercie très sincèrement Marta Sanz-Solé et Josselin Garnier d'avoir accepté de rapporter cette thèse, pour leur lecture attentive de ce manuscrit et pour leurs commentaires.

Je tiens aussi à exprimer ma gratitude à Sylvie Méléard, François Coquet et Ying Hu, qui par ailleurs dirige l'équipe Processus Stochastiques de l'IRMAR où je suis rattaché, de s'être intéressés à ce travail et d'avoir accepté d'être membres de mon jury. Je suis aussi très reconnaissant à Anne de Bouard des discussions que nous avons eues et de m'avoir invité au congrès de la SMAI qui m'a permis de discuter des applications physiques. Je la remercie également de faire partie de ce jury.

J'adresse de très sincères remerciements à Michel Pierre pour m'avoir accueilli au plateau de mathématiques de l'antenne de Bretagne de l'ENS Cachan où il est toujours très agréable de venir travailler. Je le remercie aussi pour sa gentillesse et ses encouragements.

Ma gratitude va aussi à Christian Robert qui a bien voulu m'accueillir dans son équipe de Statistique au CREST et pour plusieurs discussions fructueuses sur des travaux encore en cours lors de la rédaction de cette thèse.

Je tiens à remercier les personnes de l'INSEE qui m'ont permis de mener cette thèse conjointement à mon travail quotidien à l'Unité de Méthodologie Statistique et tout particulièrement Pascal Ardilly, Daniel Verger et Marc Christine. Je salue au passage mes collègues et amis de l'INSEE.

Toute ma reconnaissance va également à mes différents professeurs de mathématiques à l'antenne de Bretagne de l'ENS Cachan, aux universités de Rennes 1, d'Orsay et d'avant. Ils ont su me transmettre leur goût pour les mathématiques et leurs applications qui a suscité mon engagement dans cette thèse.

Je tiens également à exprimer ma grande sympathie envers mes collègues et amis de Rennes et Paris : Pierre Alquier, Patrice Bertail, Patrice Boissolles, Virginie Bonnaillie-Noël, Gabriel Caloz, Michel Delecroix, Paul Doukhan, Christophe Dupont, Emmanuelle Gautherat, Ghislaine Gayraud, Hugo Harari-Kermadec, Cyril Joutard, Nicolas Landais, Jimy Lamboley, Benoît Merlet, Mounir Rihani, Judith Rousseau, Mathieu Rosenbaum, Cyril Odasso, Karel Pravda-Starov, Rozenn Texier-Picard, Sandrine Toldo, Grégory Vial, Julien Vovelle, Jean-Michel Zakoian, mais aussi les membres du laboratoire de Processus Stochastiques.

Je souhaite par ailleurs remercier les équipes administratives toujours très disponibles.

J'ai aussi une pensée pour les baillacois, les voileux, Matthieu, Guéno, Florian et Julie, Amaury, Yassine et Natacha, Céline, Joan et Louba, Eric, Solen et Damien, Etienne, Karine et Joaquín, Vincent et Catherine, Romain, Alexandre M, Vincent et Teresa, Didier et Fabiana, Alexandre P, Emilie et Antonin et tous mes autres amis.

Enfin, j'ai une pensée toute particulière pour mes parents, Alain et Bérénice et les autres membres de ma famille qui m'ont beaucoup soutenu. Je suis aussi très reconnaissant à Anne d'avoir partagé ce travail à mes côtés et de m'avoir toujours encouragé. Je remercie également Chantal, Claude et Damien pour leur soutien.

Table des matières

1 Introduction **7**

1.1 Éléments sur les équations déterministes 7

1.1.1 Propriétés du groupe linéaire 8

1.1.2 L'équation non linéaire, caractère localement bien posé du problème de Cauchy 10

1.1.3 Invariants de l'équation, existence globale et explosion en temps fini 11

1.1.4 Les ondes solitaires 13

1.2 Les perturbations aléatoires 16

1.2.1 Des motivations 16

1.2.2 Le bruit, le processus de Wiener et les mesures Gaussiennes 20

1.2.3 Les équations de Schrödinger non linéaires stochastiques 25

1.3 Les grandes déviations 27

1.3.1 Présentation 27

1.3.2 Des résultats généraux 29

1.3.3 Transport par image directe de PGDs 30

2 Présentation des résultats **33**

2.1 Grandes déviations et support pour un bruit additif 33

2.2 Grandes déviations uniformes pour un bruit multiplicatif . . . 39

2.3 Application à la transmission par solitons 42

2.4 Application à la sortie d'un domaine d'attraction 47

2.5 Le cas d'un bruit additif fractionnaire 51

3 Conclusion et perspectives **57**

A Large deviations in the additive case and applications **61**

A.1 Introduction 61

A.2 Notations and preliminary results 64
A.2.1 Properties of the group 66
A.2.2 Topology and trajectory spaces 66
A.2.3 Statistical properties of the noise 69
A.2.4 The random perturbation 70
A.2.5 Continuity with respect to the perturbation 73
A.3 Sample path large deviations 75
A.4 The support of the law of the solution 78
A.5 Applications to the blow-up times 79
A.5.1 Probability of blow-up after time T 80
A.5.2 Probability of blow-up before time T 82
A.5.3 Bounds for the approximate blow-up time 82
A.6 Applications to nonlinear optics 84
A.6.1 Upper bounds 88
A.6.2 Lower bounds 89

B Large deviations in the multiplicative case **95**
B.1 Introduction 95
B.2 Preliminaries and statement of the result 97
B.3 Continuity of the skeleton with respect to the control on the sets C_a and exponential tail estimates 101
B.4 Almost continuity of the Itô map 109
B.5 End of the proof of the uniform LDP 114
B.6 Applications to the blow-up times 116

C Application to the timing jitter **119**
C.1 Introduction 119
C.2 Notations and preliminaries 124
C.3 Tails of the the mass and center with additive noise 132
C.4 Tails of the center in the multiplicative case 142
C.5 Annex - proof of Theorem C.2.1 146

D Exit from a domain for weakly damped equations **149**
D.1 Introduction 149
D.2 Preliminaries 152
D.3 Exit from a domain of attraction in L^2 156
D.4 Exit from a domain of attraction in H^1 170
D.5 Annex - proof of Theorem D.2.1 180

E The case of a fractional additive noise **185**
E.1 Introduction 185
E.2 Preliminaries 187
E.3 Local well posedness and necessary results 192
E.4 The main results 197

Chapitre 1

Introduction

1.1 Éléments sur les équations déterministes

Les équations de Schrödinger non linéaires (NLS) apparaissent dans de nombreux modèles de la physique. Elles interviennent pour décrire l'évolution d'enveloppes de trains d'ondes monochromatiques de la forme

$$\epsilon\Psi \exp\left(i\mathbf{k}\cdot\mathbf{x}-\omega t\right),$$

où $\mathbf{k}$ est le vecteur d'onde, dans un milieu faiblement non linéaire et dispersif. L'amplitude est supposée petite, *i.e.* $\epsilon << 1$.

On peut citer le cas de la propagation d'un faisceau laser dans un milieu diélectrique non linéaire où l'indice de réfraction est fonction de l'amplitude de l'onde. Ceci est le cas par exemple dans une fibre optique. Dans le cas d'un milieu de Kerr où les fluctuations de l'indice de réfraction sont proportionnelles à $|\mathbf{E}|^2$, $\mathbf{E}$ étant le champ électrique polarisé dans une direction $\mathbf{e}$, il est possible d'obtenir une équation de Schrödinger non linéaire à partir des équations de Maxwell. Dans ce cas, nous parlons aussi de non linéarité cubique. L'équation décrit l'évolution de l'amplitude complexe $\mathcal{E}$ d'une onde plane monochromatique

$$\mathcal{E}\exp\left(ikz-\omega t\right)\mathbf{e}$$

où le vecteur unitaire $\mathbf{e}$ est orthogonal au vecteur d'onde $\mathbf{k}=(0,0,k)$.

Les équations apparaissent aussi pour décrire la propagation d'ondes à la surface libre d'un fluide parfait dans un canal infini de profondeur infinie ou dans un plasma. Par exemple elles décrivent, par exemple, l'évolution de la phase superfluide de l'hélium II au zéro absolu. Dans ce cas, les équations peuvent s'obtenir à partir d'équations des ondes non linéaires en utilisant notamment un développement autour de l'onde monochromatique et une analyse multi-échelle. On trouve aussi ces équations en mécanique quantique en localisant le potentiel d'intéraction V de l'équation de Hartree

$$i\frac{\partial u}{\partial t} + \frac{1}{2}\Delta u + \left(V * |u|^2\right) u = 0$$

ou en physique des solides pour certaines structures moléculaires comme approximation continue de systèmes de molécules ou d'atomes en intéraction disposés sur un réseau (chaînes anharmoniques d'atomes, agrégats moléculaires de Scheibe...).

D'une manière générale, ces équations s'écrivent sous la forme

$$i\frac{\partial u}{\partial t} = \mathrm{div}\,(\mathcal{D}\nabla u) + f(x, |u|^2)u.$$

Nous nous restreindrons au cas où la matrice $\mathcal{D}$ vaut l'identité, l'opérateur non borné est alors le Laplacien, et au cas où la non linéarité est de la forme $f(x, |u|^2)u = \lambda |u|^{2\sigma} u$ où $\lambda = \pm 1$. Nous considérons l'équation dans tout l'espace $\mathbb{R}^d$.

1.1.1 Propriétés du groupe linéaire

Considérons ici l'équation linéaire

$$\frac{\partial u}{\partial t} = -i\Delta u$$

appelée aussi équation libre.

Introduisons les espaces de Hilbert de fonctions à valeurs complexes L^2 ou H^s, $s \geq 0$. L'espace de Lebesgue L^2 des fonctions de carré intégrable est muni du produit scalaire défini pour u et v dans L^2 par $(u, v)_{\mathrm{L}^2} = \mathfrak{Re} \int_{\mathbb{R}^d} u(x)\overline{v}(x)$. L'espace de Sobolev H^s est l'espace des distributions tempérées u dont la transformée de Fourier $\hat{u}$ satisfait $(1 + |\xi|^2)^{s/2}\hat{u} \in \mathrm{L}^2$. L'opérateur $-i\Delta u$ est un opérateur non borné sur les espaces H^s. Il est défini sur un domaine, par exemple $D(-i\Delta u) = \mathrm{H}^2$ dans le cas d'un opérateur dans L^2. Les espaces L^p sont les espaces de Lebesgue standard et les espaces $W^{k,p}$ sont les espaces de Sobolev de fonctions de L^p ayant des dérivées partielles jusque l'ordre k dans L^p.

L'opérateur étant anti auto-adjoint, il est, d'après le théorème de Stone, le générateur infinitésimal d'un groupe unitaire $(U(t))_{t\in\mathbb{R}}$ fortement continu.

Les opérateurs $U(t)$ sur l'espace considéré, L^2 ou H^1, sont des isométries. Ce groupe a une forme explicite pour des fonctions de la classe de Schwartz, il s'agit de la convolution par le noyau

$$K(t,x) = \frac{1}{(4i\pi t)^{\frac{d}{2}}} \exp\left(-\frac{i|x|^2}{4t}\right),$$

c'est à dire

$$\forall t \neq 0, \forall u_0 \in \mathcal{S}(\mathbb{R}^d),\ U(t)u_0 = K(t,\cdot) * u_0$$

Le groupe ne possède pas de propriété de régularisation globale. Il admet par contre des propriétés de régularisation locale.

Le groupe possède des propriétés d'intégrabilité que nous utiliserons très fréquemment. Tout d'abord, nous avons des estimées de décroissance :

$$\forall p \geq 2,\ \forall t \neq 0,\ \forall u_0 \in \mathrm{L}^{p'}(\mathbb{R}^d),\ \|U(t)u_0\|_{\mathrm{L}^p(\mathbb{R}^d)} \leq (4\pi|t|)^{-d\left(\frac{1}{2}-\frac{1}{p}\right)} \|u_0\|_{\mathrm{L}^{p'}(\mathbb{R}^d)}.$$

Nous observons qu'en particulier nous avons, pour $p > 2$,

$$\lim_{t\to\pm\infty} \|U(t)u_0\|_{\mathrm{L}^p(\mathbb{R}^d)} = 0.$$

Ainsi, bien que le groupe sur L^2 soit une isométrie, nous avons la décroissance vers 0 de la norme L^∞ et de toutes les normes L^p entre 2 et ∞. Une solution même localisée au départ "s'étale" au cours du temps. Il s'agit d'une propriété de dispersion. Nous pouvons aussi exprimer cette propriété autrement. Considérons une onde plane monochromatique $u(t,x) = \exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t)$ et injectons cette fonction dans l'équation linéaire. Nous obtenons la relation de dispersion réelle $\omega = |k|^2$ et nous avons $\det D_k\omega \neq 0$. Dans ce cas, les ondes de vecteurs d'ondes différents se propagent à des vitesses différentes. Il en résulte un étalement du paquet d'ondes.

Rappelons désormais les inégalités de Strichartz.
Une paire $(r(p),p)$ est dite admissible si $2 \leq p < \frac{2d}{d-2}$ lorsque $d > 2$ ($2 \leq p < +\infty$ si $d = 2$ ou $2 \leq p \leq +\infty$ si $d = 1$) et

$$\frac{2}{r(p)} = d\left(\frac{1}{2} - \frac{1}{p}\right).$$

La première inégalité de Strichartz, voir par exemple [25, 99], donne un résultat d'intégrabilité du flot linéaire. Elle s'énonce

(i) il existe C positif tel que pour u_0 dans L^2, T positif et $(r(p), p)$ paire admissible,

$$\|U(\cdot)u_0\|_{\mathrm{L}^{r(p)}(0,T;\mathrm{L}^p)} \leq C\,\|u_0\|_{\mathrm{L}^2}.$$

La seconde inégalité de Strichartz donne des propriétés d'intégrabilité de la "convolution" par le groupe linéaire. Cette convolution apparaît lorsque nous considérons l'équation non linéaire, l'équation non linéaire stochastique, ou une équation non homogène, comme perturbations de l'équation linéaire ; c'est ce que nous faisons lorsque nous étudions des solutions mild. Ces solutions d'équations aux dérivées partielles (EDP) s'expriment sous une forme analogue à celle donnée par la méthode de la variation de la constante pour les équations différentielles ordinaires (EDO), dans ce cas on parle aussi de forme de Duhamel. La deuxième inégalité de Strichartz s'écrit

(ii) Pour tout T positif, $(r(p), p)$ et $(r(q), q)$ des paires admissibles, s et ρ tels que $\frac{1}{s}+\frac{1}{r(q)}=1$ et $\frac{1}{\rho}+\frac{1}{q}=1$, il existe C positif tel que pour f dans $\mathrm{L}^s\,(0,T;\mathrm{L}^\rho)$,

$$\left\|\int_0^{\cdot} U(\cdot-s)f(s)ds\right\|_{\mathrm{L}^{r(p)}(0,T;\mathrm{L}^p)} \leq C\|f\|_{\mathrm{L}^s(0,T;\mathrm{L}^\rho)}.$$

Notons que les exposants p et q peuvent être choisis de manière complètement indépendante, ceci résultant d'un argument de dualité.

1.1.2 L'équation non linéaire, caractère localement bien posé du problème de Cauchy

L'équation non linéaire est traitée comme une perturbation de l'équation linéaire. Nous nous intéressons à des solutions faibles ou de manière équivalente à des solutions mild. Celles-ci satisfont pour t positif

$$u(t) = U(t)u_0 - i\lambda\int_0^t U(t-s)|u(s)|^{2\sigma}u(s)ds$$

où u_0 est la donnée initiale que nous prenons dans L^2 ou dans H^1 pour des valeurs de σ convenables. Dans le cas d'une équation dans H^1 avec donnée initiale dans H^1 et une dimension d'espace, on peut vérifier, grâce aux inégalités de Hölder et de Gagliardo-Nirenberg, que la non linéarité est Lipschitzienne sur les bornés de H^1 quelque soit σ. Sinon, la non linéarité n'est pas localement Lipschitzienne et l'inégalité de Strichartz (ii) nous permet de donner un sens à la "convolution" avec le groupe. On procède dans tous les cas en appliquant le théorème du point fixe de Picard à l'application

$$\mathfrak{F}_{u_0}: \; u \mapsto U(t)u_0 - i\lambda \int_0^t U(t-s)|u(s)|^{2\sigma}u(s)ds.$$

Les espaces complets dans lesquels on applique le point fixe dès que $d > 1$ ou dans L^2 sont des intersections d'espaces de Banach, c'est à dire l'intersection ensembliste munie du maximum des normes des deux espaces de Banach intersectés. Ils sont définis respectivement pour des données initiales dans L^2 et pour des données initiales dans H^1 par

$$Y^{(T,p)} = \mathrm{L}^r(0,T;\mathrm{L}^p) \cap \mathrm{C}\left(0,T;\mathrm{L}^2\right),$$

$$X^{(T,p)} = \mathrm{L}^r\left(0,T;\mathrm{W}^{(1,p)}\right) \cap \mathrm{C}\left(0,T;\mathrm{H}^1\right).$$

On peut alors par exemple montrer, voir [25, 99, 136], que le problème est localement bien posé dans H^1 quel que soit σ si $d \leq 2$ et $\sigma < \frac{2}{d-2}$ si $d > 2$. Ainsi, les solutions existent en temps petit, dépendant de la donnée initiale. En effet, en temps suffisamment petit, l'application $\mathfrak{F}_{u_0}$ est contractante sur une boule invariante de l'espace complet, voir [25, 99].

1.1.3 Invariants de l'équation, existence globale et explosion en temps fini

On peut vérifier que l'équation est invariante par un certain nombre de transformations : invariance par translation de la phase, invariance par translation temporelle, invariance par translation spatiale, par rotation spatiale, par transformation Galiléenne, invariance d'échelle et invariance pseudo-conforme (appelée aussi invariance par transformation de lentille) en dimension critique. L'équation possède par conséquent, d'après le théorème de Noether, un certain nombre de quantités invariantes parmi lesquelles la masse

$$\mathbf{N}\left(u(t)\right) = \int_{\mathbb{R}^d} |u(t,x)|^2 dx,$$

l'Hamiltonien

$$\mathbf{H}\left(u(t)\right) = \frac{1}{2}\int_{\mathbb{R}^d} \left|\nabla u(t,x)\right|^2 dx - \frac{\lambda}{2\sigma+2}\int_{\mathbb{R}^d} |u(t,x)|^{2\sigma+2} dx,$$

le moment linéaire

$$\mathbf{P}\left(u(t)\right) = 2\mathfrak{Re}\left(i\int_{\mathbb{R}^d} \overline{u(t,x)}\nabla u(t,x)dx\right),$$

le moment angulaire

$$\mathbf{M}\left(u(t)\right) = 2\mathfrak{Re}\left(i\int_{\mathbb{R}^d} x \wedge \left(\overline{u(t,x)}\nabla u(t,x)\right)dx\right).$$

Le flot de l'EDP est Hamiltonien, on peut réécrire l'équation sous la forme

$$\frac{\partial u}{\partial t} = i\frac{\delta \mathbf{H}\left(u\right)}{\delta u} = \frac{\delta \mathbf{H}\left(u\right)}{\delta \overline{u}}$$

où $\mathrm{d}\mathbf{H}(u) \cdot h = \left(\frac{\delta \mathbf{H}(u)}{\delta u}, h\right)$.

Citons également les deux relations qui suivent.
Le centre de masse, ou position, défini par la relation $\mathbf{Y}\left(u(t)\right) = \int_{\mathbb{R}^d} x|u(t,x)|^2 dx$ satisfait l'équation

$$\frac{\mathrm{d}\mathbf{Y}\left(u(t)\right)}{\mathrm{d}t} = \mathbf{P}\left(u(t)\right).$$

Le centre de masse est aussi souvent défini en divisant la quantité précédente par la masse totale en t, rappelons que celle-ci est ici constante.
La variance, définie par $\mathbf{V}\left(u(t)\right) = \int_{\mathbb{R}^d} |x|^2|u(t,x)|^2 dx$, satisfait la relation appelée identité de la variance, ou théorème du viriel,

$$\frac{\mathrm{d}^2\mathbf{V}\left(u(t)\right)}{\mathrm{d}t^2} = 8\mathbf{H}\left(u(t)\right) - 4\frac{d\sigma - 2}{\sigma + 1}q\int_{\mathbb{R}^d} |u(t,x)|^{2\sigma+2}dx.$$

L'invariance de la masse entraîne que les solutions existent globalement dans L^2. Dans le cas H^1, on obtient, grâce à l'invariance de la masse et de l'Hamiltonien, que le problème de Cauchy est bien globalement posé dès que $\sigma < \frac{2}{d}$, cas d'une non linéarité sous-critique. Lorsque la non linéarité est critique ou sur-critique ,$\sigma = \frac{2}{d}$ et $\frac{2}{d} < \sigma < \frac{2}{d-2}$, le problème est globalement bien posé lorsque $\lambda = -1$, cas de la non linéarité répulsive ou défocalisante. Si $\lambda = 1$ et $\frac{2}{d} \leq \sigma < \frac{2}{d-2}$, si $d \geq 3$, ou si $\lambda = 1$ et $\frac{2}{d} \leq \sigma$, si $d = 1$ ou $d = 2$, les solutions peuvent exploser en temps fini. Cette propriété correspond en physique à un transfert énergétique violent des grandes aux petites échelles. Le premier résultat rigoureux est dû à Glassey exploite l'identité de la variance, voir [87]. Plusieurs résultats existent, citons en particulier

Theorem 1.1.1 *Pour une donnée initiale $u_0 \in \mathrm{H}^1(\mathbb{R}^d)$ telle que $\mathbf{V}\left(u_0\right) < \infty$ et $\sigma \geq \frac{d}{2}$, il existe $t_* < \infty$ tel que*

$$\begin{cases} \lim_{t\to t_*} \|\nabla u(t)\|_{\mathrm{L}^2(\mathbb{R}^d)} = \infty \\ \lim_{t\to t_*} \|u(t)\|_{\mathrm{L}^\infty(\mathbb{R}^d)} = \infty \end{cases}$$

si l'une des conditions suivantes est satisfaite :
(i) $\mathbf{H}(u_0) < 0$,
(ii) $\mathbf{H}(u_0) = 0$ et $\mathfrak{Im} \int_{\mathbb{R}^d} x.\overline{u_0}(x)\nabla u_0(x)dx < 0$,
(iii) $\mathbf{H}(u_0) > 0$ et $\mathfrak{Im} \int_{\mathbb{R}^d} x.\overline{u_0}(x)\nabla u_0(x)dx \leq -4\sqrt{\mathbf{H}(u_0)}\|xu_0\|_{\mathrm{L}^2(\mathbb{R}^d)}$.

Theorem 1.1.2 *Si $d = 1$ et $\sigma = 2$, toute condition initiale de* $\mathrm{H}^1(\mathbb{R}^d)$ *telle que* $\mathbf{H}(u_0) < 0$ *explose en temps fini.*
Si $d \geq 2$ et $\frac{2}{d} \leq \sigma \leq \frac{2}{d-2} \wedge 2$, toute condition initiale de $\mathrm{H}^1(\mathbb{R}^d)$ *à symétrie radiale, i.e. ne dépendant que de $|x|$, telle que* $\mathbf{H}(u_0) < 0$, *explose en temps fini.*

Ce deuxième résultat dû à Ogawa et Tsutsumi présente l'intérêt de s'affranchir de la restriction à des données initiales de variance finie. Il existe également un certains nombre de résultats fins, voir [136], sur l'explosion : taux d'explosion, auto-similarité et concentration de la masse en dimension critique...

1.1.4 Les ondes solitaires

Un équilibre entre effet non linéaire et dispersion peut entraîner l'existence de solutions que l'on appelle ondes solitaires ou, dans certains cas, solitons. Ce phénomène a été découvert empiriquement par l'ingénieur John Scott Russel en 1834. Se promenant sur le bord d'un canal, il vit se déplacer une onde qui s'était formée à la proue d'un bateau tiré par des chevaux. Cette onde se mit à cheminer seule, alors que le bateau s'était arrêté, sur une longue distance, sans que sa forme ou sa vitesse ne s'altère. L'EDP décrivant la propagation de vagues de grande longueur d'onde et de petite amplitude A à la surface d'un canal de faible profondeur h_0 est l'équation de Korteweg-de Vries. Elle peut s'obtenir à partir des équations d'Euler. Elle possède des solutions de type ondes progressives de la forme $\varphi(x - ct)$. Dans le cas d'une dimension en espace, la fonction φ s'écrit

$$\varphi(z) = A\mathrm{sech}^2\left(\sqrt{\frac{3A}{4h_0^3}}z\right),$$

la vitesse est donnée par $c = \sqrt{gh_0}\left(1 + \frac{A}{2h_0}\right)$. Rappelons que la fonction sécante hyperbolique s'exprime $\mathrm{sech}(z) = \frac{1}{\cosh(z)}$. La solution est donc localisée.

Les ondes solitaires interviennent désormais dans plusieurs branches de la physique : en optique, en physique des plasmas, en astrophysique, en hydrodynamique... La vague de mascaret qui remonte le long de certains fleuves est en général suivie d'un train de solitons. Des ondes solitaires se propagent aussi sur la couche thermocline de l'océan et sont engendrées par la topographie du sol. Des ondes solitaires atmosphériques existent et se manifestent sous la forme du nuage "Morning Glory" en Australie. Celui-ci se développe en présence d'une inversion de température et peut être engendré par une activité orageuse ou une collision entre des fronts de brise océanique opposés. Les solitons ont aussi été introduits en physique des solides (pour certaines chaînes d'atomes, certains des cristaux) et en biologie (cinétique des réactions biologiques, dénaturation thermique de l'ADN).

Des ondes solitaires existent pour l'équation de Schrödinger non linéaire lorsque la non linéarité est attractive ou focalisante, *i.e.* le cas où $\lambda = 1$. Ce ne sont plus cette fois des solutions onde progressive mais plutôt des états stationnaires, c'est à dire des solutions de la forme $\exp(i\omega t)\varphi(x)$ où ω est strictement positif. La terminologie stationnaire vient de la mécanique quantique. En effet, l'intensité de l'onde est invariante par translation temporelle ce qui correspond à la probabilité de trouver la particule à un endroit particulier.

Le profil φ satisfait l'EDP

$$\Delta\varphi - \omega\varphi + |\varphi|^{2\sigma}\varphi = 0, \tag{1.1.1}$$

la condition $\omega > 0$ assure que $\lim_{|x|\to\infty} \varphi(x) = 0$ c'est à dire que les solutions sont localisées. Cette équation admet une unique solution positive lorsque la dimension d'espace vaut $d = 1$, elle est donnée par

$$\varphi(z) = (\omega(\sigma+1))^{\frac{1}{2\sigma}} \operatorname{sech}^{\frac{1}{\sigma}}\left(\sqrt{\omega}\sigma z\right).$$

Les fonctions φ sont aussi les points critiques dans H^1 de la fonctionnelle de Lyapunov appelée aussi fonctionnelle d'action

$$\mathcal{S}(u) = \frac{1}{2}\left\{\mathbf{H}(u) + \omega\mathbf{N}(u)\right\}.$$

En effet, l'EDP peut se réécrire

$$\frac{\delta\mathcal{S}(u)}{\delta u} = 0.$$

Nous verrons, *c.f.* annexe D, que le problème d'optimisation dual intervient lorsque l'on cherche à caractériser le point de sortie d'un niveau d'énergie pour des équations de Schrödinger non linéaires avec un petit bruit additif et un amortissement faible.

Nous avons le résultat qui suit.

Theorem 1.1.3 *Supposons $d \geq 2$ et $\sigma < \frac{2}{d-2}$ (pour tout σ si $d = 2$) alors il existe une unique solution g positive, à symétrie radiale, appartenant à $\mathrm{C}^2(\mathbb{R}^d)$, à l'équation (1.1.1). En outre g et ses dérivées jusque l'ordre 2 admettent une décroissance exponentielle à l'infini. Cette solution minimise l'action $\mathcal{S}$ parmi toutes les fonctions de H^1. Par contre, il existe une infinité de solutions de classe $\mathrm{C}^2(\mathbb{R}^d)$ à symétrie radiale.*

La solution g est appelée état fondamental ou "ground state".

Les ondes solitaires peuvent posséder des propriétés particulièrement intéressantes. Par exemple, dans le cas intégrable, la rencontre de deux solitons de l'équation de Korteweg-de Vries se traduit par l'absence de déformation de forme et par un déphasage. On peut alors faire une analogie avec l'intéraction de deux particules. De plus, ces solutions possèdent en général des propriétés de stabilité orbitale. Dans le cas de l'équation de Schrödinger non linéaire, nous avons le résultat qui suit.

Theorem 1.1.4 *Si $\sigma < \frac{2}{d}$ et g est l'état fondamental, alors, pour tout ϵ positif, il existe η positif tel que, si la donnée initiale u_0 satisfait*

$$\inf_{\theta\in\mathbb{R},y\in\mathbb{R}^d} \|u_0(\cdot) - \exp(i\theta)g(\cdot + y)\|_{\mathrm{H}^1} < \eta,$$

alors la solution u du problème de Cauchy est telle que

$$\inf_{\theta\in\mathbb{R},y\in\mathbb{R}^d} \|u(t,\cdot) - \exp(i\theta)g(\cdot + y)\|_{\mathrm{H}^1} < \epsilon, \quad \forall t > 0.$$

L'ensemble stable est une variété de dimension infinie qui tient compte des groupes laissant invariant l'équation. Les ondes solitaires sont donc des ondes que l'on pourrait qualifier de robustes. Des résultats de stabilité asymptotique existent également. Ils valent pour des données initiales particulières, reposent sur des hypothèses sur le linéarisé de l'opérateur non linéaire au voisinage de l'onde solitaire... Des résultats d'instabilité pour des non linéarités critiques et sur-critiques existent également, voir [136].

Par extension dans le cas où $\sigma = d = 1$, modèle en optique, nous appellerons aussi solitons les solutions non stationnaires

$$u(t,x) = \sqrt{2}A\mathrm{sech}(A(x-x_0)+2A\Omega t)\exp\left(-i(A^2-\Omega^2)t + i\Omega(x-x_0) + i\theta_0\right).$$

Dans ce cas nous appelons Ω la vitesse de groupe et θ_0 la phase initiale. Le centre de telles ondes se déplace alors à la vitesse constante $\frac{\mathrm{d}\mathbf{Y}(u(t))}{\mathrm{d}t} = -8\Omega A$. Lorsque la vitesse de groupe est nulle ces solutions restent centrées en 0.

Du point de vue de l'optique, Hasegawa et Tappert ont proposé en 1973 d'utiliser les solitons de l'équation de Schrödinger non linéaire pour coder des bits et transmettre sur de très longues distances des données à très haut débit. Les solitons de l'équation de Schrödinger non linéaires permettent aussi de rendre compte des transferts d'énergie dans les structures moléculaires, par exemple les protéines, les molécules d'ADN ou la photosynthèse. En effet, dans le cas d'un potentiel d'intéraction entre plus proches voisins avec un terme harmonique et un terme d'ordre 4, en postulant des oscillations internes et des solutions solitons, les physiciens obtiennent que la limite continue de l'équation de l'enveloppe du soliton est l'équation de Schrödinger non linéaire. Enfin, il semblerait également que les solitons de l'équation de Schrödinger non linéaire modélisent les vagues scélérates à la surface de l'océan.

1.2 Les perturbations aléatoires

Nous allons nous intéresser à des perturbations aléatoires des équations précédentes. Un terme aléatoire supplémentaire est ajouté, il peut s'agir d'une force, terme supplémentaire de type $\dot{\eta}(t,x)$, ou d'un potentiel, terme de type $u\dot{\eta}(t,x)$. Nous parlons de bruit additif ou multiplicatif. Il est paramétré par le temps et les variables d'espaces. Les équations bruitées que nous considérons sont motivées par la physique. En optique ou en physique des solides (cristaux, chaînes d'atomes de type Fermi Pasta Ulam, agrégats moléculaires de Scheibe) le bruit peut être additif ou multiplicatif.

1.2.1 Des motivations

Dans de nombreuses modélisations le bruit est introduit afin de rendre compte du fait que l'on ne peut pas décrire parfaitement le système à partir des grandeurs à disposition. Ceci est le cas par exemple en économie où une partie des grandeurs est inobservable. Les systèmes évoluant de façon déterministe au court du temps, et pour un temps "continu", sont souvent décrits par une EDO ou une EDP si l'évolution temporelle est liée à des fluctuations spatiales. On peut alors rendre compte de l'incertitude sur l'état du système, par exemple l'évolution d'une particule en présence de fluctuation de la température ou la valeur d'un produit financier, par un terme de bruit dans l'équation. Nous sommes alors en présence d'équations différentielles stochastiques (EDS) ou aux dérivées partielles stochastiques (EDPS). Prenons le cas des équations aux dérivées partielles de type équations d'évolutions. Ces équations sont souvent valables sur une certaine plage de valeurs des paramètres. Elles décrivent l'évolution dans des milieux idéalisés et par exemple ne tiennent pas compte des impuretés et des fluctuations des propriétés du milieu notamment en fonction des fluctuations de la température. Elles correspondent aussi à des approximations de modèles plus complexes et négligent des termes d'ordres plus élevés. Le terme aléatoire peut alors rendre compte des termes que l'on a négligés et/ou des fluctuations du milieu. De ce point de vue, il est intéressant d'évaluer la robustesse des résultats qualitatifs obtenus pour ces modèles idéalisés en présence d'une petite perturbation. Nous allons voir par la suite que le bruit peut aussi être tout à fait intrinsèque.

Le système peut aussi être excité par des forces extérieures que l'on n'arrive pas à décrire. Il peut s'agir, par exemple, d'une impulsion électrique sur un neurone, voir [139]. Celle ci arrive à des intervalles de temps irréguliers et à divers endroits du neurone. En optique des fibres l'équation de Schrödinger non linéaire apparaît comme modèle. Un signal permet de coder un 1, l'absence de signal un 0. Un tel signal est en fait amorti, ceci empêche la propagation sur de longues distances, typiquement plus de 1000 km. Plusieurs types d'amplifications permettent de palier à ce problème. Considérons dans un premier temps les amplificateurs par dopage à l'Erbium. Dans ce cas, des chaînes d'amplificateurs régulièrement espacés le long de la fibre permettent de compenser l'amortissement. Le bruit est alors tout à fait intrinsèque au phénomène d'amplification. Du point de vue de la mécanique quantique, la mesure d'une quantité physique se fait nécessairement avec une certaine imprécision et il ne peut pas exister de mesure optique de l'intensité lumineuse sans incertitude. De ce fait montrons qu'un amplificateur qui ne présenterait pas d'émission spontanée de bruit violerait le principe d'incertitude d'Heisenberg. Celui-ci stipule qu'il existe une limite fondamentale à la mesure simultanée du moment p et de la position x d'une particule. Les incertitudes étant notées Δp et Δx satisfont

$$\Delta p \Delta x \geq \frac{\hbar}{2}$$

où $\hbar = \frac{h}{2\pi}$ est la constante de Planck. Dans notre cas, les particules sont des photons. En supposant qu'ils se propagent le long de l'axe des x, nous obtenons alors la borne suivante sur l'incertitude de la mesure conjointe de l'énergie E du photon et de son temps d'arrivée t

$$\Delta E \Delta t \geq \frac{\hbar}{2}.$$

En effet, nous avons $p = \hbar k = \frac{\hbar\omega}{c} = \frac{E}{c}$, k étant le nombre d'onde et c la célérité de la lumière, et $\Delta t = \frac{\Delta x}{c}$. Supposons alors que l'incertitude sur l'énergie soit liée à l'incertitude sur le nombre de photons, la fréquence ν étant elle parfaitement déterminée. L'incertitude sur l'énergie vaut alors $\Delta E = h\nu\Delta n$ et celle sur la phase $\Delta\varphi = 2\pi\nu\Delta t$. Nous obtenons alors la borne d'incertitude nombre-phase

$$\Delta n\Delta\varphi \geq \frac{1}{2}.$$

Supposons qu'un amplificateur sans émission spontanée de bruit existe et qu'il soit de gain G. Soit $n_0 \pm \Delta n_0$ le nombre de photons incidents, $\varphi_0 \pm \Delta\varphi_0$ la phase du signal d'entrée, $n \pm \Delta n = G\,(n_0 \pm \Delta n_0)$ le nombre de photons à la sortie, et $\varphi_\pm\Delta\varphi = \varphi_0 \pm \Delta\varphi_0 + \theta$ la phase du signal de sortie ; la translation de phase θ rend compte de la propagation du signal dans l'amplificateur. Supposons que nous mesurions le signal après l'amplificateur avec un détecteur pour lequel l'incertitude est minimale alors nous avons la relation $\Delta n\Delta\varphi = \frac{1}{2}$. Cette incertitude correspond à l'incertitude $\Delta n_0\Delta\varphi_0 = \frac{1}{2G} < \frac{1}{2}$ sur le signal incident avant amplification. Ceci contredit le principe d'incertitude d'Heisenberg. On peut aussi montrer, voir [52], que l'amplitude minimale P_N du bruit qui permette de respecter le principe d'incertitude d'Heisenberg à l'entrée et à la sortie de l'amplificateur est donnée par

$$P_N = \frac{h\nu(G-1)}{2T} = h\nu B(G-1),$$

T représente l'intervalle de temps sur lequel le détecteur mesure le signal et B la largeur de bande de l'amplificateur. Une description quantique du bruit est possible mais devient beaucoup plus complexe, voir en particulier [52]. Il est par contre remarquable que, même si le bruit a des origines quantiques, il n'est pas nécessaire de rentrer précisément dans les principes de la mécanique quantique afin de décrire l'effet du bruit sur la transmission d'un signal. En général, on modélise pour ce type d'amplification le bruit comme un bruit additif, voir par exemple [50, 51, 62, 64, 65]. Ceci est aussi le cas dans le premier exemple du neurone.

Il existe aussi d'autres types d'amplification, *c.f.* [52]. Citons des amplificateurs paramétriques.
L'amplification de Raman dans une fibre monomode a été suggérée en 1989-1992. Le processus physique qui rend possible l'amplification de Raman est l'intéraction de la lumière avec les phonons optiques. Dans ce cas, l'évolution

du nombre de photons n est régie par l'équation différentielle

$$\pm\frac{\mathrm{d}n}{\mathrm{d}x} = g_R \frac{P_p}{A_{eff}}(n+1) - \alpha n$$

où g_R est le gain de Raman, $\frac{P_p}{A_{eff}}$ correspond à l'intensité injectée et α à la perte de la fibre.
Le mélange paramétrique de quatre ondes ou mélange stimulé de 4 photons est étudié pour la première fois en 1974-1975. Il est appliqué en 1980-1985 pour l'amplification d'ondes progressives. Ces amplificateurs sont un autre type d'amplificateurs paramétriques.
Ces deux types d'amplification sont aussi accompagnés d'émission spontanée de bruit. Dans les deux cas, le bruit apparaît sous forme multiplicative, voir [59, 98] pour l'amplification de Raman et [101] pour le processus de mélange de quatre ondes. L'amplification de Raman contribue aussi à la nonlinéarité de Kerr ; il alors figure également dans l'équation un terme de réponse de Raman supplémentaire, *c.f.* [98].

Dans le cas des systèmes de communication optiques, prendre en compte le bruit doit permettre de quantifier la dégradation du signal depuis la source d'émission et le récepteur à l'extrémité de la fibre. Plus particulièrement, à l'extrémité de la fibre, un récepteur converti la lumière en courant électrique. Les photons incidents sont capturés et un électron est émis, simultanément une transition entre états liés a lieu. Le signal électrique est préamplifié puis intégré. Un circuit électronique mesure alors l'énergie sur une fenêtre qui s'exprime en fonction de la période inter émission des bits et la compare à un niveau de référence. Ceci permet de décider si le signal est un 1 ou un 0. Le taux d'erreur sur les bits (BER pour bit-error rate) s'exprime

$$BER = \mathbb{P}(1|0)p(0) + \mathbb{P}(0|1)p(1)$$

où $\mathbb{P}(1|0)$ est la probabilité conditionnelle que le circuit électronique décide de manière erronée qu'un 1 a été émis alors qu'un 0 était émis (on définit de manière analogue $\mathbb{P}(0|1)$) ; $p(0)$ et $p(1)$ sont les probabilités de transmettre les symboles 0 et 1. Lorsque le nombre de bits du message est élevé on peut supposer que $p(0) = p(1) = \frac{1}{2}$. L'obtention du BER nécessite de connaître les probabilités conditionnelles et donc la loi de la quantité mesurée en l'extrémité de la fibre en fonction de la donnée initiale. L'approximation Gaussienne est souvent utilisée en pratique et cette pratique est critiquée par plusieurs physiciens. Mais aussi, nous sommes intéressés par l'influence de la longueur de la fibre et des paramètres liés à l'émission du signal (amplitude, période inter émission...) sur le BER. Il est donc souhaitable que, dans ce domaine, la prise en compte du bruit permette de répondre à ces question pratiques.

Enfin, notons qu'il existe aussi beaucoup d'autres sources d'aléa dans les équations de Schrödinger non linéaires. Tout d'abord le signal est produit par un faisceau laser et la donnée initiale est connue de manière imprécise, toujours d'après le principe d'incertitude d'Heisenberg. Mais aussi, d'autres termes peuvent être sujets à des fluctuations aléatoires, il peut s'agir de la non linéarité, du potentiel, de la dispersion, d'une biréfringence ou encore d'un potentiel quadratique, voir par exemple [1, 76, 78].

Nous avons ici surtout développé les motivations de l'introduction d'un terme de bruit dans les équations de Schrödinger nonlinéaires stochastiques issues de l'optique non linéaire. Le bruit apparaît aussi dans les modèles de transferts d'énergie dans les structures moléculaires. Il permet de rendre compte des fluctuations thermiques. Le bruit peut être additif dans le cas d'excitons qui créent ou absorbent des photons, *c.f.* [11, 16], ou multiplicatif lorsqu'il est admis que les transferts d'énergie se font sans création ni perte de masse, *c.f.* [10, 11, 12]. L'introduction d'un terme de bruit dans ces équations intervient aussi dans la modélisation de l'évolution d'un condensat de Bose-Einstein.

1.2.2 Le bruit, le processus de Wiener et les mesures Gaussiennes

En dimension 1, un bruit blanc correspond à une collection de variables aléatoires $(X_j)_{j\in J}$ centrées, de même variance σ^2 et non corrélées entre elles. Dans $\mathbb{R}^d$, les variables aléatoires sont des vecteurs aléatoires centrés, de même matrice de variance-covariance Σ (éventuellement différente de l'identité I) et non corrélés entre eux. Si J indice le temps, lorsque $\Sigma \neq I$ le bruit est dit blanc en temps et coloré en espace au sens où pour un temps $j \in J$ les composantes sont corrélées. Si le bruit n'est pas dégénéré, *i.e.* $\det \Sigma \neq 0$, un bruit blanc en temps et coloré en espace est l'image via $\Sigma^{\frac{1}{2}}$ d'un bruit blanc en temps et en espace. Les bruits que nous considérons, hormis dans le dernier article de l'annexe E, sont blancs en temps à valeurs dans un espace de dimension infinie. Ils seront également colorés en espace ; cette propriété est inhérente à la dimension infinie et apparaît pour des raisons de sommabilité. Précisons désormais le lien entre bruit blanc et mouvement Brownien.

Rappelons qu'un mouvement Brownien est un processus $(\beta_t)_{t\in\mathbb{R}}$ à trajectoires continues issu de 0, à accroissements indépendants (en particulier c'est une martingale et un processus de Markov) et tel que pour $t > s$, $\frac{\beta_t - \beta_s}{\sqrt{t-s}}$ suit la loi $\mathcal{N}(0,1)$. La dernière propriété est une propriété de stationnarité. De manière équivalente, il s'agit d'un processus Gaussien (les marginales fini-dimensionnelles $(\beta_{t_1}, ..., \beta_{t_n})$ sont des vecteurs Gaussiens), centré, de fonction de covariance $\mathbb{E}(B_t B_s) = t \wedge s$. Le mouvement Brownien est lui, contrairement au bruit blanc, corrélé en temps ; comme cela est précisé plus haut, il s'agit d'un processus de Markov. La mesure de Wiener est la mesure image sur $\mathrm{C}([0,T])$ de la probabilité $\mathbb{P}$ sur l'espace probabilisé sous-jacent Ω par l'application qui à $\omega \in \Omega$ associe la trajectoire $t \mapsto B_t(\omega)$.

Considérons le bruit blanc à valeurs dans $\mathbb{R}$, notons $S_n = \sum_{i=1}^n X_i$ la marche aléatoire ($S_0 = 0$) et définissons la trajectoire interpolée $S_t = S_{\lfloor t \rfloor} + (t - \lfloor t \rfloor) X_{\lfloor t \rfloor + 1}$. Le théorème de Donsker indique qu'à changement d'échelle et, à la renormalisation correspondant à celle du théorème de la limite centrale près, la marche aléatoire interpolée converge en loi vers une trajectoire Brownienne. Il donne plus précisément que la loi du processus $\left(S_t^n = \frac{1}{\sigma\sqrt{n}} S_{nt}\right)_{t \geq 0}$, où $\sigma^2 = \mathbb{E}(X_i^2)$, converge étroitement vers la mesure de Wiener.

Par ailleurs, le mouvement Brownien définit une variable aléatoire à valeurs dans l'espace de Banach $\mathrm{C}([0,T])$ ou dans $\mathrm{L}^2(0,T)$. L'opérateur de covariance Q dans $\mathrm{L}^2(0,T)$, muni du produit scalaire usuel $(\cdot, \star)_{\mathrm{L}^2}$ s'exprime en fonction de la fonction de covariance par, étant donné φ et ψ dans $\mathrm{L}^2(0,T)$,

$$(Q\varphi, \psi)_{\mathrm{L}^2} = \int_0^T (t \wedge s)\varphi(s)ds.$$

L'opérateur est auto-adjoint compact et nous avons, en corollaire du théorème usuel de diagonalisation des opérateurs auto-adjoints compacts, la décomposition de Karhunen-Loeve :
il existe une famille $(\xi_j)_{j \in \mathbb{N}}$ de variables aléatoires de loi normale standard $\mathcal{N}(0,1)$ indépendantes et $(e_j)_{j \in \mathbb{N}}$ une base Hilbertienne de $\mathrm{L}^2(0,T)$ constituée de fonctions propres de Q associées aux valeurs propres λ_j telles que

$$\beta(t) = \sum_{j \in \mathbb{N}} \sqrt{\lambda_j} \xi_j e_j(t).$$

Dans le cas où $T = 1$, nous avons

$$\beta(t) = \sqrt{2} \sum_{j \in \mathbb{N}} \xi_j \frac{\sin\left(\left(n + \frac{1}{2}\right)\pi t\right)}{\left(n + \frac{1}{2}\right)\pi}.$$

Bien qu'à priori non sommable, la dérivée formelle de la série ci-dessus est la somme d'une série à coefficients aléatoires où chaque composante d'une seconde base de $\mathrm{L}^2(0,T)$ est indépendante et de loi $\mathcal{N}(0,\sigma^2)$. Plus précisément, le bruit Gaussien blanc en temps continu est la dérivée au sens des distributions du mouvement Brownien. Il s'agit ainsi, même en dimension infinie, d'un vecteur dont toutes les composantes sont indépendantes et de loi $\mathcal{N}(0,\sigma^2)$. L'intégrale stochastique permet de donner un sens sous forme intégrale à des EDOs perturbées par un bruit blanc.

Nous considérerons dans ce qui suit des mouvements Brownien W à trajectoires dans un espace de Hilbert ou plus généralement de Banach, appelés processus de Wiener. Dans la première définition du mouvement Brownien, il suffit de remplacer $\frac{\beta_t-\beta_s}{\sqrt{t-s}}$ suit la loi $\mathcal{N}(0,1)$ par $\frac{W_t-W_s}{\sqrt{t-s}}$ a pour loi une mesure Gaussienne centrée μ.

Rappelons qu'une mesure de probabilité μ sur un espace de Banach réel séparable E, muni de sa tribu Borélienne, est une mesure Gaussienne si pour tout élément e^* du dual topologique E^*, la forme linéaire continue $< e^*, \cdot >_{E^*,E}$ sur E, où le crochet est celui de la dualité E^*-E, définit une variable aléatoire réelle centrée. La mesure de Wiener sur $\mathrm{C}([0,T])$ est une mesure Gaussienne. Le théorème de Fernique donne en particulier l'existence de moments de tout ordre.

Introduisons la notion d'espace de Hilbert noyau auto-reproduisant, voir par exemple [23] pour une introduction. Il s'agit d'un espace d'énergie lié à la covariance, il caractérise la mesure Gaussienne centrée. Nous verrons que cet espace intervient dans la fonction de taux d'un principe de grandes déviations pour des familles de mesures Gaussiennes et pour caractériser leurs supports. Définissons par R l'application

$$\begin{array}{rcl} R: E^* & \to & E \\ e^* & \mapsto & R(e^*) = \int_E < e^*, e >_{E^*,E} e \ \mu(de) \end{array}$$

l'intégrale est une intégrale de Bochner, elle est bien définie comme un élément de E, d'après le théorème de Fernique. L'espace de Hilbert, noyau auto-reproduisant H_μ de la mesure Gaussienne centrée μ sur E, est le complété de $R(E^*)$ pour le produit scalaire

$$\langle R(e^*), R(f^*)\rangle_{H_\mu} = \int_E < e^*, e >_{E^*,E} < f^*, e >_{E^*,E} \mu(de)$$

Rappelons aussi que H_μ s'injecte continûment dans E (l'injection sera notée i) et que cette image est de mesure nulle pour la mesure μ. Le support d'une telle mesure Gaussienne μ est donné par *supp* $\mu = \overline{H_\mu}^E$, *c.f.* [8]. Signalons aussi que pour toute base Hilbertienne $\left(e_j^*\right)_{j\in\mathbb{N}}$ de H_μ^*, base duale de $(e_j)_{j\in\mathbb{N}}$, et e dans E, nous avons, d'après le théorème de convergence des martingales,

$$\lim_{n\to\infty}\sum_{j=0}^{n} < e_j^*, e >_{E^*,E} e_j = e.$$

Une telle décomposition en série permet, par exemple, de définir l'intégrale stochastique par rapport à un processus de Wiener sur E à partir de l'intégrale stochastique usuelle par rapport au mouvement Brownien. Enfin, nous avons la représentation en diagramme

$$E^* \overset{i^*}{\hookrightarrow} H_\mu^* \overset{R}{\longleftrightarrow} H_\mu \overset{i}{\hookrightarrow} E.$$

L'application R est en fait l'identification de Riesz entre le dual topologique H_μ^* et H_μ. Ce formalisme est celui des espaces de Wiener abstraits.

Considérons le cas de la loi d'un processus Gaussien à trajectoires continues sur $[0,1]$. Le dual topologique de $\mathrm{C}([0,1])$ étant l'espace des mesures signées sur $[0,1]$, nous avons, d'après le théorème de Fubini, pour t appartenant à $[0,1]$

$$\begin{aligned} Re^*(t) = \delta_t(Re^*) \quad &= \textstyle\int_E < e^*, e >_{E^*,E} e(t)\mu(de) \\ &= \textstyle\int_E \left(\int_0^1 e(s)e^*(ds)\right) e(t)\mu(de) \\ &= \textstyle\int_0^1 \left(\int_E e(s)e(t)\mu(de)\right) e^*(ds) \\ &= \textstyle\int_0^1 \gamma(s,t)e^*(ds) \end{aligned}$$

où $\gamma(s,t)$ est la fonction de covariance. Dans le cas particulier de la mesure de Wiener nous obtenons

$$Re^*(t) = \int_0^t e^*([s,1])ds$$

et

$$\langle Re^*, Rf^*\rangle_{H_\mu} = \int_0^1 (Re^*)'(t)\,(Rf^*)'(t)dt.$$

L'espace de Hilbert noyau auto-reproduisant est l'espace de Sobolev $\mathrm{H}_0^1(0,1)$. Il est constitué de fonctions absolument continues, nulles en 0. On l'appelle aussi espace de Cameron-Martin.

Lorsque l'espace E est un espace de Hilbert réel séparable H muni du produit scalaire $(\cdot,\star)_H$, la mesure μ admet un opérateur de covariance Q sur H satisfaisant pour tout h et k dans H

$$(Qh,k)_H = \int_H (h,l)_H(k,l)_H\mu(dl) = \mathbb{E}\left[(h,\cdot)_H(k,\cdot)_H\right].$$

Dans ce cas on peut montrer que l'espace de Hilbert noyau auto-reproduisant est isométrique à $\mathrm{Im}Q^{\frac{1}{2}}$ muni de la structure image.

Etant donné un espace de Hilbert réel séparable H, considérons la famille $(\mu_{e_1,\dots,e_n})$ des mesures Gaussiennes centrées de dimension finie et de covariance l'identité indexée par les familles orthonormales de H sur les tribus cylindriques

$$\sigma\left\{h \in H:\ ((e_1,h)_H,...,(e_n,h)_H) \in \mathcal{B}(\mathbb{R}^n)\right\},$$

où $\mathcal{B}(\mathbb{R}^n)$ est la tribu Borélienne de $\mathbb{R}^n$. Elle définit sur la réunion des tribus cylindriques une fonction d'ensembles μ. Cette fonction est appelée mesure cylindrique. Néanmoins, la réunion des tribus cylindriques n'est pas une tribu et μ n'est pas $\sigma-$additive. Si H et H' sont des espaces de Hilbert et si Φ est un opérateur linéaire continu de H dans H' et μ une mesure cylindrique sur H alors l'image directe $\Phi_*\mu$ est $\sigma-$additive (et donc peut être étendue en une mesure de Radon) si et seulement si Φ est Hilbert-Schmidt. Lorsque H' est un espace de Banach, nous pouvons introduire la notion d'opérateur $\gamma-$radonifiant, voir en particulier [17, 18].

Un processus de Wiener cylindrique W_c sur un espace de Hilbert réel séparable H est tel que, quelle que soit $(e_j)_{j\in\mathbb{N}}$ une base Hilbertienne de H, il existe une suite $(\beta_j)_{j\in\mathbb{N}}$ de mouvements Browniens indépendants telle que $W_c = \sum_{j\in\mathbb{N}}\beta_j e_j$. Ce processus n'est pas bien défini, en particulier ses marginales, $W_c(t)$ pour t positif, n'admettent pas de moment d'ordre 2. Par contre son image directe par un opérateur Φ Hilbert-Schmidt ou $\gamma-$radonifiant est bien définie. Si Φ est à valeurs un espace de Hilbert, la covariance du processus image vaut $Q = \Phi\Phi^*$. D'après ce qui précède, un processus de Wiener W de covariance Q sur un espace de Hilbert réel séparable H est l'image directe, via l'injection Hilbert-Schmidt de $\mathrm{Im}Q^{\frac{1}{2}}$ dans H, d'un processus de Wiener cylindrique sur le noyau auto-reproduisant de la loi de $W(1)$. Comme mentionné plus haut, l'intégrale stochastique par rapport à un processus de Wiener est définie via l'intégrale stochastique usuelle dans $\mathbb{R}$ grâce à la décomposition en série, voir [34] pour un cadre Hilbert. Le bruit considéré dans nos équations aux dérivées partielles stochastiques est la dérivée en temps, au sens des distributions, d'un processus de Wiener. La décomposition formelle de la dérivée en temps d'un processus de Wiener cylindrique est la somme d'un développement en série sur le produit tensoriel des bases de $L^2(0,T)$ et de H dont les coefficients sont une collection dénombrable de variables aléatoires $\mathcal{N}(0,\sigma^2)$ indépendantes. Nous appelons ce bruit un bruit blanc espace-temps. Dans le cas d'un processus de Wiener usuel, le bruit est blanc en temps et coloré en espace.

Il existe également une seconde approche tout à fait équivalente du calcul stochastique permettant de donner un sens aux EDP stochastiques *c.f.* [79, 126]. Cette approche s'appelle parfois approche variationnelle. Elle est aussi assez répandue, voir par exemple [22, 29, 139]. Sous cette approche, dans le cas d'un domaine borné, nous prenons par exemple $[0,1]$, et d'un temps dans $[0,T]$, un bruit blanc W d'intensité λ, la mesure de Lebesgue, est une fonction sur les éléments de la tribu $\mathcal{B}$ des Boréliens de $[0,1]\times[0,T]$ telle que :

(i) $W(A)$ est une variable aléatoire $\mathcal{N}(0,\lambda(A))$

(ii) si $A\cap B=\emptyset$ alors $W(A)$ et $W(B)$ sont indépendants
et $W(A\cup B)=W(A)+W(B)$ *p.s.*

La condition (ii) signifie que W est une fonction additive d'ensembles. Le champ Gaussien $W_{x,t}=W\left([0,x]\times[0,t]\right)$ est appelé drap Brownien ; il correspond aussi à $\int_0^x W_c(t,x)dx$ où W_c est un processus de Wiener cylindrique sur $L^2(0,1)$. L'intégrale stochastique dans ce cadre consiste à définir une intégrale par rapport à des mesures martingales.

1.2.3 Les équations de Schrödinger non linéaires stochastiques

Nous étudions dans cette thèse deux types d'équations de Schrödinger non linéaires stochastiques. Une avec bruit additif et une avec un bruit multiplicatif particulier.
L'équation avec bruit additif s'écrit sous forme d'Itô

$$i\mathrm{d}u-\left(\Delta u+\lambda|u|^{2\sigma}u\right)\mathrm{d}t=\mathrm{d}W$$

où W est un processus de Wiener sur L^2 ou H^1 selon que nous étudions des solutions dans L^2 ou dans H^1.
L'équation avec bruit multiplicatif s'écrit quant à elle

$$i\mathrm{d}u-\left(\Delta u+\lambda|u|^{2\sigma}u\right)\mathrm{d}t=u\circ\mathrm{d}W$$

où $\circ$ désigne le produit Stratonovich et W est un processus de Wiener sur $\mathrm{L}^2_{\mathbb{R}} \cap \mathrm{L}^\alpha$ ou $\mathrm{H}^1_{\mathbb{R}} \cap \mathrm{W}^{1,\alpha}$ selon que nous étudions des solutions dans L^2 ou dans H^1. L'indice $\mathbb{R}$ signifie que nous considérons des espaces de fonctions à valeurs dans $\mathbb{R}$. La condition sur α est $\alpha > 2d$. Lorsqu'on définit W comme ΦW_c avec Φ $\gamma-$radonifiant de H (par exemple L^2) dans $\mathrm{L}^2_{\mathbb{R}} \cap \mathrm{L}^\alpha$ ou $\mathrm{H}^1_{\mathbb{R}} \cap \mathrm{W}^{1,p}$, nous pouvons réécrire le produit Stratonovich en fonction du produit Itô, via un terme de tendance supplémentaire qui correspond au terme de crochet. Nous obtenons

$$i\mathrm{d}u - \left(\Delta u + \lambda |u|^{2\sigma} u - \frac{i}{2} u F_\Phi\right) \mathrm{d}t = u \mathrm{d}W,$$

où $F_\Phi(x) = \sum_{j\in\mathbb{N}} (\Phi e_j(x))^2$ pour x dans $\mathbb{R}^d$ et $(e_j)_{j\in\mathbb{N}}$ est une base Hilbertienne de H.

Notons que pour certaines EDPS particulières, en particulier pour l'équation de la chaleur en dimension 1, il est possible d'étudier le bruit blanc. Ce sont des équations où le semi-groupe linéaire a des propriétés de régularisation globale. Dans ce cas, dans la forme mild, le semi-groupe S qui apparaît dans la convolution stochastique, par exemple $\int_0^t S(t-s)dW(s)$ dans le cas d'un bruit additif, possède lui-même la propriété d'être Hilbert-Schmidt dès que $t > s$. Ici le groupe est une isométrie sur les espaces de Hilbert basés sur L^2, il ne possède pas de telles propriétés. Notons qu'en physique les auteurs étudient généralement le cas de perturbations aléatoires d'équations de Schrödinger nonlinéaires de type bruit blanc. Nous n'arrivons pas à donner de sens mathématique au bruit blanc dans ce cas. Une limite bruit blanc, dans un sens qui sera précisé, est par contre considérée dans les sections 2.1 et 2.3.

Il est prouvé dans [36, 37] que le problème de Cauchy est localement bien posé dans L^2 ou dans H^1 pour des exposants σ suffisamment petits. Les solutions sont des solutions faibles au sens de l'analyse des équations aux dérivées partielles. Ce sont de manière équivalente des solutions mild. Dans le cas du bruit additif, la condition sur σ est la même que dans le cas déterministe. Dans le cas du bruit multiplicatif, la condition sur σ est la même que dans le cas déterministe si $d = 1$ ou $d = 2$. Sinon, dans le cas L^2, la condition est $\sigma < \frac{2}{d} \wedge \frac{1}{d-1}$ et, dans le cas H^1, $\sigma < 2$ si $d = 3$ et $\frac{1}{2} \leq \sigma < \frac{2}{d-2}$ ou $\sigma < \frac{1}{d-1}$ si $d \geq 4$. Par ailleurs, le bruit étant réel et le produit étant un produit Stratonovich, la masse est conservée. Cette propriété est motivée par la physique. Le cadre H^1 est alors adapté à l'étude de l'explosion en temps finie. En effet, on ne peut pas observer le phénomène d'explosion dans L^2, en tout cas pour un bruit multiplicatif, car la masse est conservée. Dans ce cas, lorsque le problème est localement bien posé, il est aussi globalement bien posé. Pour les équations stochastiques dans H^1, avec bruit additif ou multiplicatif, les solutions sont globales pour des non linéarités sous-critiques ou dans le cas défocalisant dès que le problème est localement bien posé, *c.f.* [37]. Ce résultat coincide avec celui pour les équations déterministes.

Les équations ont aussi été étudiées d'un point de vue numérique dans [13, 43]. Dans [43], le cas de la dimension 1 est considéré. L'influence des deux types de bruit sur la propagation des solitons et sur l'explosion en temps fini est étudiée. Il est obtenu qu'un bruit additif a tendance à accélérer l'explosion en temps fini alors que le bruit multiplicatif a tendance à la retarder. Mais aussi, il est obtenu que toute donnée initiale explose en temps fini dans les cas critiques et sur-critiques. Le cas du bruit blanc est aussi étudié. Il apparaît que le bruit blanc multiplicatif empêche l'explosion en temps fini. Dans [13], le cas de la dimension 2 est étudié. L'effet du bruit sur l'explosion en temps fini est étudié d'un point de vue théorique dans [38, 39, 40] pour des bruits additifs complexes et réels et des bruits multiplicatifs. Sous des hypothèses légèrement plus fortes sur la régularité du processus de Wiener, il est montré que, pour des non linéarités sur-critiques et des données initiales non nulles, quel que soit t strictement positif, la probabilité que la solution explose avant t est strictement positive.

1.3 Les grandes déviations

1.3.1 Présentation

Les résultats de grandes déviations permettent de quantifier une loi faible des grands nombres. Supposons qu'une famille de mesures de probabilités $(\mu^\epsilon)_{\epsilon>0}$, sur un espace de Banach muni de sa tribu Borélienne, converge étroitement vers une mesure de Dirac en un point x, alors nous savons que, pour tout Borélien A ne contenant pas x dans son intérieur, $\lim_{\epsilon\to 0}\mu^\epsilon(A) = 0$. Dans le langage des probabilités un tel évènement A est un évènement de grandes déviations. La déviation est grande car l'ensemble A ne dépend pas de ϵ. Un résultat de grandes déviations quantifie la convergence vers 0 à vitesse ϵ. L'espace de Banach dans cette thèse sera généralement un espace de trajectoires ou la droite réelle. Cela pourrait aussi être un espace de mesures, par exemple dans le cas de mesures empiriques, nous parlons dans ce cas de grandes déviations de niveau 2.

Le cas le plus élémentaire est celui où $(X_j)_{j\geq 1}$ est une famille de variables aléatoires réelles, centrées, indépendantes, et de même loi, et où on s'intéresse à quantifier la convergence en loi vers 0 de la variable aléatoire $\overline{S}_n = \frac{1}{n}\sum_{j=1}^n X_j$. Si la loi des variables aléatoires est $\mathcal{N}(0,1)$, la loi de $\overline{S}_n$ est encore Gaussienne et nous pouvons vérifier aisément que

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(|\overline{S}_n| \geq \delta\right) = -\frac{\delta^2}{2}.$$

Le théorème de Cramer donne que ce résultat est encore valable dans le cas non Gaussien. La limite peut ne plus être définie, le résultat donne malgré tout un encadrement des limites inférieures et supérieures. Cet encadrement dépend de la loi des variables aléatoires X_j via une fonctionnelle que nous appelons fonction de taux. Nous appelons cela un principe de grandes déviations. Le théorème de Cramer s'énonce de la façon qui suit.

Theorem 1.3.1 *La famille des lois μ_n des variables aléatoires $\overline{S}_n$ satisfait un principe de grandes déviations (PGD) de vitesse $\frac{1}{n}$ et de fonction de taux Λ^*, la transformée de Fenchel-Legendre, définie par*

$$\Lambda^*(x) = \sup_{\lambda\in\mathbb{R}} \{\lambda x - \Lambda(\lambda)\}$$

où $\Lambda(\lambda)$ est le logarithme de la transformée de Laplace de la loi de X_1. En d'autres termes, pour tout Borélien A de $\mathbb{R}$, nous avons la suite d'inégalités

$$-\inf_{x\in \overset{\circ}{A}} \Lambda^*(x) \leq \underline{\lim}_{n\to\infty} \frac{1}{n} \log \mu_n(A) \leq \overline{\lim}_{n\to\infty} \frac{1}{n} \log \mu_n(A) \leq -\inf_{x\in \overline{A}} \Lambda^*(x)$$

Cet énoncé nous montre qu'un principe de grandes déviations transforme un problème de calcul des probabilités en un problème de minimisation. Dans cette thèse nous serons confrontés à des problèmes de contrôle optimal voire de calcul des variations. Mais aussi, un problème de minimisation peut trouver une réponse probabiliste. Par ailleurs, nous voyons qu'un résultat de grandes déviations fait intervenir la topologie et que les bornes supérieures et inférieures sont d'autant meilleures que la topologie est fine. Enfin, la fonction de taux est une fonctionnelle semi-continue inférieurement. Une bonne fonction de taux I est une fonction de taux telle que l'ensemble des niveaux inférieurs à un réel strictement positif c, *i.e.* $I^{-1}([0,c])$, est compact. Cette propriété assure que, sur de tels ensembles, la fonction de taux atteint son minimum.

Dans un espace polonais E (nous notons par B ses boules), nous pouvons réécrire de façon équivalente les bornes supérieures et inférieures d'un PGD

pour une famille de mesures $(\mu^\epsilon)_{\epsilon>0}$, de vitesse ϵ et de bonne fonction de taux I.
La borne supérieure prend la forme :
quels que soient c et δ strictement positifs, $K(c)^\delta$ un $\delta-$voisinage de $I^{-1}([0,c])$, nous avons

$$\overline{\lim}_{\epsilon\to 0}\epsilon \log \mu^\epsilon\left(\left(K(c)^\delta\right)^c\right) \leq -c.$$

La borne inférieure quant à elle se réécrit sous la forme :
quels que soient e dans E et δ strictement positif, nous avons

$$\underline{\lim}_{\epsilon\to 0}\epsilon \log \mu^\epsilon\left(B(e,\delta)\right) \geq -I(e).$$

Nous appelons cette écriture écriture à la Freidlin-Wentzell.

1.3.2 Des résultats généraux

Le théorème de Cramer s'étend à $\mathbb{R}^d$ et à la dimension infinie. Des résultats de grandes déviations existent aussi pour des suites de variables non i.i.d., c'est le cas du théorème de Gärtner-Ellis ou des résultats de grandes déviations pour les chaînes de Markov, *c.f.* [48]. Nous présentons désormais dans cette section deux résultats de grandes déviations qui nous serons utiles par la suite.

Du théorème de Cramer en dimension infinie nous pouvons déduire un résultat général de grandes déviations pour les mesures Gaussiennes sur les espaces de Banach réels séparables. Considérons un tel espace de Banach E, une mesure Gaussienne μ sur E, d'espace de Hilbert noyau auto-reproduisant H_μ, et la famille de mesures $(\mu_\epsilon)_{\epsilon>0}$ images directes de μ par la transformation $x \mapsto \sqrt{\epsilon}x$ sur E. Nous avons le résultat qui suit, *c.f.* [53].

Theorem 1.3.2 *La famille $(\mu_\epsilon)_{\epsilon>0}$ satisfait un PGD sur E, de vitesse ϵ et de bonne fonction de taux la transformée de Fenchel-Legendre Λ_μ^*. Par ailleurs la transformée de Fenchel-Legendre Λ_μ^* se réécrit*

$$\Lambda_\mu^*(e) = \begin{cases} \frac{1}{2}\|e\|_{H_\mu}^2 & si\ e \in H_\mu \\ \infty & si\ e \in E \setminus H_\mu \end{cases}.$$

Nous pouvons déduire de ce résultat le théorème de Schilder qui énonce un PGD pour les trajectoires du mouvement Brownien.

Theorem 1.3.3 *La famille des lois des trajectoires des processus* $(\sqrt{\epsilon}\beta_t)_{t\in[0,1]}$ *sur* $\mathrm{C}([0,1])$ *satisfait un PGD de vitesse* ϵ *et de bonne fonction de taux*

$$I(f) = \begin{cases} \frac{1}{2}\int_0^1 |f'(t)|^2 dt & si\ f \in H_0^1(0,1) \\ \infty & si\ f \notin H_0^1(0,1) \end{cases}.$$

Soit J un ensemble partiellement ordonné, filtrant à droite (c'est à dire qu'étant donné i et j dans J, il existe k dans J tel que $i \leq k$ et $j \leq k$). Un système projectif est la donnée de $(\mathcal{Y}_j, p_{i,j})_{j\in J}$ où les espaces $\mathcal{Y}_j$ sont des espaces topologiques séparés et les applications p_{ij} de $\mathcal{Y}_i$ dans $\mathcal{Y}_j$ sont continues et satisfont $p_{ik} = p_{ij} \circ p_{jk}$ dès que $i \leq j \leq k$. La limite projective $\mathcal{X}$ de ce système est le sous espace de l'espace produit $\prod_{j\in J} \mathcal{Y}_j$ constitué des éléments (y_j) tels que $y_i = p_{ij}(y_j)$, muni toujours de la topologie produit. Les projections $p_j : \mathcal{X} \to \mathcal{Y}_j$ sont continues. Le théorème de Dawson-Gartner donne un principe de grandes déviations pour des mesures sur un espace limite projective.

Theorem 1.3.4 *Soit* $(\mu_\epsilon)_{\epsilon>0}$ *une famille de mesures de probabilité sur* $\mathcal{X}$ *telles que pour tout* j *dans* J *les mesures* $(p_j)_*\mu_\epsilon$ *sur* $\mathcal{Y}_j$ *satisfont un PGD de vitesse* ϵ *et de bonne fonction de taux* I_j. *Alors la famille* $(\mu_\epsilon)_{\epsilon>0}$ *satisfait un PGD de vitesse* ϵ *et de bonne fonction de taux*

$$I(x) = \sup_{j\in J} I_j(p_j(x)), \quad x \in \mathcal{X}.$$

Ce troisième théorème permet aussi de montrer le théorème de Schilder.

1.3.3 Transport par image directe de PGDs

Les PGDs peuvent se transporter par image directe et image réciproque. Remarquons que l'image directe est adaptée au transport de la compacité. Commençons par rappeler le principe de contraction de Varadhan.

Theorem 1.3.5 *Soient* $\mathcal{X}$ *et* $\mathcal{Y}$ *deux espaces vectoriels topologiques séparés et une fonction* f *de* $\mathcal{X}$ *dans* $\mathcal{Y}$ *continue. Soit une famille* $(\mu_\epsilon)_{\epsilon>0}$ *de mesures sur* $\mathcal{X}$ *satisfaisant un PGD de vitesse* ϵ *et de bonne fonction de taux* I *alors la famille* $(f_*\mu_\epsilon)_{\epsilon>0}$ *satisfait un PGD sur* $\mathcal{Y}$ *de vitesse* ϵ *et de bonne fonction de taux*

$$I'(y) = \inf_{x\in\mathcal{X}:\ y=f(x)} I(x).$$

Il est cependant possible d'imposer une propriété plus faible que la continuité pour f. Nous aurons besoin de résultats de ce type pour montrer un PGD pour les lois des trajectoires des équations de Schrödinger non linéaires stochastiques lorsque le bruit est multiplicatif. En effet, la démarche générale pour montrer un PGD au niveau des trajectoires des équations de Schrödinger non linéaires stochastiques est de transférer par image directe un PGD pour le processus de Wiener, résultat du type du théorème de Schilder pour le mouvement Brownien. Voici une première approche, cela ne sera pas celle adoptée dans le présent papier. Cette méthode peut permettre de montrer un PGD pour les trajectoires d'une EDS (voir par exemple [48, 66]).

Theorem 1.3.6 *Soit $(\mu^\epsilon)_{\epsilon>0}$ une famille de mesures de probabilité satisfaisant un PGD de bonne fonction de taux I sur un espace topologique séparé $\mathcal{X}$. Soit également une suite de fonctions continues $(f_j)_{j\in\mathbb{N}}$ de $\mathcal{X}$ dans un espace métrique $(\mathcal{Y}, d)$. Supposons qu'il existe une application f de $\mathcal{X}$ dans $\mathcal{Y}$ telle que*

$$\forall c > 0,\ \overline{\lim}_{j\to\infty} \sup_{x\in\mathcal{X}:\ y=f(x)} d\left(f_m(x), f(x)\right) = 0.$$

Alors, toute famille de mesures de probabilité $(\tilde{\mu}^\epsilon)_{\epsilon>0}$ telle que

$$\lim_{\mathrm{J}\to\infty} \overline{\lim}_{\epsilon\to 0} \epsilon \log \mathbb{P}^{\epsilon,j}\left(\Gamma_\delta\right) = -\infty, \tag{1.3.1}$$

où $\mathbb{P}^{\epsilon,j}$ est le produit tensoriel de $(f_j)_* \mu^\epsilon$ et de $\tilde{\mu}_\epsilon$ et

$$\Gamma_\delta = \left\{(y,z) \in \mathcal{Y}^2 :\ d(y,z) > \delta\right\},$$

satisfait un PGD sur $\mathcal{Y}$ de vitesse ϵ et de bonne fonction de taux

$$I'(y) = \inf_{y\in\mathcal{Y}:\ y=f(x)} I(x).$$

La relation (1.3.1) signifie que les mesures $\left((f_j)_* \mu^\epsilon\right)_{\epsilon>0}$ sont des approximations exponentiellement bonnes des mesures $(\tilde{\mu}^\epsilon)_{\epsilon>0}$.

Nous utiliserons une autre extension du principe de contraction de Varadhan, celle-ci utilise le lemme d'Azencott appelé aussi inégalité de Freidlin-Wentzell. Elle sera présentée dans le chapitre qui suit. Notons aussi que, dans le cas des PGDs pour les trajectoires de nos EDPS, nos familles de mesures sont paramètrées par la donnée initiale. Nous nous intéresserons aussi à des PGDs uniformes, en un sens qui sera précisé, en la donnée initiale. On peut aussi trouver dans [131] une extension du principe de contraction de Varadhan permettant de transporter des PGDs uniformes pour une formulation à la Freidlin-Wentzell.

Chapitre 2

Présentation des résultats

Nous présentons dans cette partie les résultats des différents articles que rassemble cette thèse. Les articles figurent en annexe. Nous étudions les équations de Schrödinger non linéaires stochastiques et considérons la limite lorsque le bruit tend vers 0. Nous prouvons donc des principes de grandes déviations au niveau des lois des trajectoires des solutions. Ils sont énoncés dans des espaces de trajectoires explosives et pour des topologies relativement fines rendant compte des propriétés d'intégrabilité du groupe de Schrödinger. Nous donnons des applications à l'asymptotique des temps d'explosion et aux fluctuations de la masse et de la position d'un signal de type soliton. Ce dernier problème trouve par exemple des applications à l'évaluation des erreurs de transmission par solitons dans les fibres optiques. Nous donnons aussi une application au temps moyen de sortie d'un domaine et aux points de sortie dans le cas d'équations faiblement amorties. Nous étudions aussi le cas de bruits additifs plus généraux de type bruits fractionnaires en dimension infinie. Nous prouvons aussi pour des bruits additifs des théorèmes de support dans les espaces considérés.

2.1 Grandes déviations et théorèmes de support pour des équations de Schrödinger non linéaires stochastiques avec bruit additif

Nous présentons dans ce paragraphe un principe de grande déviations au niveau des trajectoires et un théorème de support ainsi que leurs applications dans le cas d'une équation de type NLS perturbée par un bruit additif. Le lecteur trouvera plus de détails ainsi que des preuves dans l'annexe A qui reprend l'article [81] publié dans *ESAIM : Probability and Statististics*.

Nous nous intéressons aux équations

$$i\mathrm{d}u^\epsilon - (\Delta u^\epsilon + \lambda |u^\epsilon|^{2\sigma} u^\epsilon)\mathrm{d}t = \sqrt{\epsilon}\mathrm{d}W, \quad \lambda = \pm 1,\ x \in \mathbb{R}^d$$

dans H^1. L'exposant σ est tel que $\sigma > 0$ si $d = 1, 2$ et $\sigma < \frac{2}{d-2}$ sinon. Nous posons $W = \Phi W_c$, où Φ est Hilbert-Schmidt de L^2 dans H^1 et W_c est un processus de Wiener cylindrique sur L^2. Le paramètre ϵ est l'amplitude du bruit. Nous cherchons à quantifier la convergence faible des lois des trajectoires vers la masse de Dirac en la solution déterministe, lorsque ϵ tend vers zéro. Nous nous intéressons donc à un principe de grandes déviations trajectoriel.

Nous souhaitons dans un premier temps pouvoir traiter des non-linéarités critiques et sur-critiques où les solutions peuvent exploser en temps fini.

Par ailleurs, nous savons que, un PGD s'énonçant pour A borélien de l'espace des trajectoires

$$-\inf_{u \in \overset{\circ}{A}} I(u) \le \underline{\lim}_{\epsilon\to 0} \epsilon \log \mathbb{P}\left(u^\epsilon \in A\right) \le \overline{\lim}_{\epsilon\to 0} \epsilon \log \mathbb{P}\left(u^\epsilon \in A\right) \le -\inf_{u \in \overline{A}} I(u),$$

plus la topologie est forte plus les bornes sont précises. En outre un PGD se transporte en un PGD pour les mesures images directes par applications continues entre espaces topologiques séparés. Ainsi nous pouvons déduire d'un PGD pour une topologie forte un PGD pour une topologie plus faible à condition que celle-ci reste séparée.

Nous introduisons donc un espace de trajectoires explosives, noté $\mathcal{E}_\infty$, muni d'une topologie relativement fine exploitant les propriétés d'intégrabilité du groupe linéaire de Schrödinger. L'espace $\mathcal{E}_\infty$ est une partie de l'espace $\mathcal{E}(\mathrm{H}^1)$ des fonctions f continues à valeurs $\mathrm{H}^1 \cup \{\Delta\}$ muni de la convergence uniforme sur les intervalles $[0, T]$ où T est inférieur strictement au temps d'explosion

$$\mathcal{T}(f) = \inf\{t > 0 :\ f(t) = \Delta\}$$

et Δ est un point cimetière. L'espace $\mathrm{H}^1 \cup \{\Delta\}$ est muni de la topologie qui est engendrée par les ouverts de H^1 et les complémentaires dans $\mathrm{H}^1 \cup \{\Delta\}$ des fermés bornés de H^1. Les fonctions f de $\mathcal{E}_\infty$ vérifient en outre des propriétés d'intégrabilité sur les intervalles $[0, T]$ où $T < \mathcal{T}(f)$. Nous posons en effet si

$d > 2$

$$\mathcal{E}_\infty = \left\{ f \in \mathcal{E}(\mathrm{H}^1) : \forall\, p \in \left[2, \frac{2d}{d-2}\right),\ \forall\, T \in [0, \mathcal{T}(f)),\ f \in \mathrm{L}^{r(p)}\left(0, T; \mathrm{W}^{1,p}\right) \right\}.$$

Lorsque $d = 1$ ou $d = 2$ nous écrivons $p \in [2, \infty)$. L'espace est muni de la topologie définie pour φ_1 dans $\mathcal{E}_\infty$ par la base de voisinages suivante

$$W_{T,p,\epsilon}(\varphi_1) = \left\{ \varphi \in \mathcal{E}_\infty : \mathcal{T}(\varphi) \geq T,\ \|\varphi_1 - \varphi\|_{X^{(T,p)}} \leq \epsilon \right\}.$$

où $T < \mathcal{T}(\varphi_1)$, p est comme défini plus haut et ϵ est strictement positif. Il s'agit d'un espace topologique séparé. Si nous notons toujours par $\mathcal{T} : \mathcal{E}_\infty \to [0, \infty]$ le temps d'explosion, l'application est mesurable et semi-continue inférieurement.

Nous vérifions la continuité de l'application qui envoie la convolution stochastique Z définie par $Z(t) = \int_0^t U(t-s)dW(s)$ en la solution. Mais aussi nous vérifions que la convolution stochastiques définit bien une variable aléatoire à valeurs dans l'espace topologique considéré et a pour loi une mesure Gaussienne centrée. Nous énonçons pour les lois de $\sqrt{\epsilon}Z$ un PGD qui découle du théorème de Dawson-Gartner pour les limites projectives et du PGD abstrait pour des familles de mesures Gaussiennes. Le PGD pour les lois des trajectoires de l'équation de Schrödinger non linéaire stochastique μ^{u^ϵ} est alors déduit par contraction. Quantifier la probabilité d'un évènement de grande déviation revient à résoudre un problème de contrôle optimal, la fonctionnelle à minimiser est la fonction de taux

$$I(u) = \frac{1}{2} \inf_{h \in \mathrm{L}^2(0,\infty;\mathrm{L}^2):\ \mathbf{S}(h)=u} \left\{ \|h\|^2_{\mathrm{L}^2(0,\infty;\mathrm{L}^2)} \right\},$$

où $\inf \emptyset = \infty$ et $\mathbf{S}(h)$, appelé squelette de l'équation stochastique, est l'unique solution mild de

$$\begin{cases} i\frac{\mathrm{d}u}{\mathrm{d}t} = \Delta u + \lambda |u|^{2\sigma} u + \Phi h, \\ u(0) = u_0 \ \in \mathrm{H}^1, \end{cases}$$

c'est à dire qu'elle s'écrit sous la forme

$$u(t) = U(t)u_0 - i\lambda \int_0^t U(t-s)|u(s)|^{2\sigma} u(s) - i \int_0^t U(t-s)\Phi h(s) ds.$$

Le PGD s'énonce de la façon suivante.

Théorème 2.1.1 (PGD) *La famille de mesures de probabilité $\left(\mu^{u^\epsilon}\right)_{\epsilon \geq 0}$ sur $\mathcal{E}_\infty$ satisfait un PGD de vitesse ϵ et de bonne fonction de taux I.*

Nous prouvons ensuite dans ce cadre le théorème de support.

Théorème 2.1.2 (Théorème de support) *Le support de la loi des solutions est caractérisé par*

$$\text{supp } \mu^{u^1} = \overline{\text{Im}\mathbf{S}}^{\mathcal{E}_\infty}.$$

Nous commençons par appliquer les deux résultats ci-dessus aux temps d'explosion et obtenons les résultats qui suivent. Dans ce qui suit nous notons u_d la solution de l'équation déterministe.

Proposition 2.1.1 *Si $u_0 \in \mathrm{H}^3$ et si l'image de Φ est dense alors pour tout $T > 0$,*

$$\mathbb{P}(\mathcal{T}(u^1) > T) > 0.$$

Proposition 2.1.2 *Si $u_0 \in \mathrm{H}^3$, si l'image de Φ est dense et si $T \geq \mathcal{T}(u_d)$, où u_d est la solution de l'équation déterministe avec donnée initiale u_0, il existe $c \in [0, \infty)$ tel que*

$$\underline{\lim}_{\epsilon \to 0} \epsilon \log \mathbb{P}\left(\mathcal{T}(u^\epsilon) > T\right) \geq -c.$$

Proposition 2.1.3 *Si $T < \mathcal{T}(u_d)$,*

$$\overline{\lim}_{\epsilon \to 0} \epsilon \log \mathbb{P}\left(\mathcal{T}(u^\epsilon) \leq T\right) \leq -U^{[0,T]} < 0.$$

où $U^{[0,T]} = \frac{1}{2} \inf_{h \in \mathrm{L}^2(0,\infty;\mathrm{L}^2) : \mathcal{T}(\mathbf{S}(h)) \leq T} \left\{ \|h\|^2_{\mathrm{L}^2(0,\infty;\mathrm{L}^2)} \right\}$.

A chaque fois nous n'avons une inégalité que dans un sens, l'autre inégalité donne un résultat trivial. Des résultats pour des approximations des temps d'explosions définis par

$$\mathcal{T}_R(f) = \{t \in [0, \infty) : \ \|f(t)\|_{\mathrm{H}^1} \geq R\}$$

sont donnés en annexe A, nous obtenons en corollaire des bornes inférieures et supérieures non triviales de $\mathbb{P}\left(S < \mathcal{T}_R\left(u_\epsilon\right) \leq T\right)$ pour $S < T < \mathcal{T}_R\left(u_d\right)$ ou $\mathcal{T}_R\left(u_d\right) < S < T$.

Enfin, nous étudions dans cette annexe l'erreur de transmission par solitons dans les fibres optiques. Cette deuxième étude est prolongée dans la section 2.3 et l'annexe C. Il a été suggéré en 1973 par Hasegawa et Tappert [93], de tirer profit de la dispersion et de la non-linéarité, apparemment facteurs limitants pour transmettre des données codées par des solitons. Nous considérons le cas de l'équation cubique avec non linéarité focalisante en dimension 1, *i.e.* $\lambda = d = \sigma = 1$. Dans ce modèle le temps est la variable d'espace et la variable d'espace une variable de temps retardé. Nous noterons donc par T l'extrémité de la fibre. Un profil de soliton, du type $\sqrt{2}\mathrm{sech}(x)$, code alors un 1 et l'absence de signal un 0. Le bruit est un bruit d'émission spontané par des amplificateurs régulièrement espacés. Nous supposons que ceux ci permettent de compenser exactement la perte dans la fibre. A l'extrémité de la fibre un récepteur mesure la quantité

$$\int_{-l}^{l} |u^{\epsilon}(T,x)|^2 dx.$$

Lorsque la mesure dépasse un certain seuil on décide que la donnée initiale était un 1 sinon on décide qu'il s'agissait d'un zéro. Du fait du bruit des erreurs de transmission peuvent se produire. Il s'agit de l'erreur de transmission d'un 1 ou d'un 0. Nous évaluons ces erreurs en procédant comme si la fenêtre était infinie, *i.e.* $l = +\infty$. En d'autres termes nous évaluons l'asymptotique des queues de la masse du signal en l'extrémité de la fibre, lorsque la donnée initiale est nulle ou un profil de soliton et lorsque le bruit tend vers 0. Notons aussi que pour les physiciens deux processus sont responsables de l'erreur de transmission : la fluctuation aléatoire de la masse et celle de la position. Nous évaluons donc l'asymptotique des queues du premier processus. Les résultats découlent du PGD. Des estimées d'énergie permettent de majorer les queues. D'autre part nous cherchons des solutions contrôlées sous la forme de solitons modulés telles que la norme du contrôle soit minimale ce qui nous permet d'obtenir une minoration des queues.

Dans le cas de la donnée initiale nulle, nous obtenons la borne supérieure

Proposition 2.1.4 *Pour tout* $T > 0$, γ *dans* $(0,1)$, *et tout opérateur* Φ *dans* $\mathcal{L}_2(\mathrm{L}^2, \mathrm{H}^1)$, *l'inégalité suivante est vérifiée*

$$\overline{\lim}_{\epsilon \to 0} \epsilon \log \mathbb{P}\left(\mathbf{N}\left(u^{\epsilon}(T)\right) \geq 4(1-\gamma)\right) \leq -\frac{1-\gamma}{2T\|\Phi\|_c^2}.$$

Nous notons ci-dessus par $\|\Phi\|_c$ la norme d'opérateur continu de L^2 dans L^2. Si nous notons

$$\Psi_{\eta}(t,x) = \sqrt{2}\eta(t) \exp\left(-i\int_0^t \eta^2(s)ds\right) \mathrm{sech}(\eta(t)x)$$

et pour une partie D dans $(0,1)$,

$$\mathcal{H}_D^1 = \left\{\eta : [0,T] \to \mathbb{R}, \text{ il existe } \tilde{\gamma} \in D \text{ tel que } \eta(t) = (1-\tilde{\gamma})\left(\frac{t}{T}\right)^2 \right\}$$

et

$$\mathcal{C}_D^1 = \Big\{ h \in \mathrm{L}^2(0,T;\mathrm{L}^2) : \quad \text{il existe } \eta \in \mathcal{H}_D^1$$
$$h(t,x) = i\frac{\eta'(t)}{\eta(t)}\Psi_\eta(t,x) - i\sqrt{2}\eta'(t)\exp\left(-i\int_0^t \eta^2(s)ds\right)\eta(t)x\frac{\sinh}{\cosh^2}\left(\eta(t)x\right)\Big\}.$$

Nous obtenons

Proposition 2.1.5 *Pour tout $T > 0$, $\gamma \in (0,1)$ et D dense dans $(0,1)$, pour toute suite d'opérateurs Hilbert-Schmidt $(\Phi_n)_{n\in\mathbb{N}}$ de L^2 dans L^2 telle que pour tout $h \in \mathcal{C}_D$, $\Phi_n h$ converge vers h dans $\mathrm{L}^1\left(0,T;\mathrm{L}^2\right)$. Alors nous avons la borne supérieure, l'exposant n rappelant que Φ est remplacé par Φ_n,*

$$\underline{\lim}_{n\to\infty,\epsilon\to 0}\epsilon\log\mathbb{P}\left(\mathbf{N}\left(u^{\epsilon,n}(T)\right) \geq 4(1-\gamma)\right) \geq -\frac{2(1-\gamma)(12+\pi^2)}{9T}.$$

Pour une donnée initiale profil de soliton nous avons la borne inférieure suivante.

Proposition 2.1.6 *Pour tout $T > 0$, γ dans $(0,1)$, et tout opérateur Φ dans $\mathcal{L}_2(\mathrm{L}^2,\mathrm{H}^1)$, l'inégalité suivante est satisfaite*

$$\overline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(\mathbf{N}\left(u^\epsilon(T)\right) < 4(1-\gamma)\right) \leq -\frac{\gamma^2}{2T\|\Phi\|_c^2(1+\gamma)^2}$$

En considérant les mêmes solitons modulés pour des paramètres dans

$$\mathcal{H}_D^2 = \Big\{ \quad \eta : [0,T] \to \mathbb{R}, \text{ il existe } \tilde\gamma \in D \text{ tel que } \eta(t) = \eta_{\tilde\gamma,T}(t)$$
$$= \left(2-\tilde\gamma-2\sqrt{1-\tilde\gamma}\right)\left(\frac{t}{T}\right)^2 + 2\left(-1+\sqrt{1-\tilde\gamma}\right)\frac{t}{T}+1 \quad \Big\}$$

et les contrôles, correspondant au cas où $\Phi = I$, dans $\mathcal{C}_D^2$ associés, nous montrons la borne supérieure suivante.

Proposition 2.1.7 *Pour tout $T > 0$, $\gamma \in (0,1)$ et D dense dans $(0,1)$, pour toute suite d'opérateurs Hilbert-Schmidt $(\Phi_n)_{n\in\mathbb{N}}$ de L^2 dans L^2 telle que pour tout $h \in \mathcal{C}_D$, $\Phi_n h$ converge vers h dans $\mathrm{L}^1\left(0,T;\mathrm{L}^2\right)$. Alors nous avons la borne supérieure, l'exposant n rappelant que Φ est remplacé par Φ_n,*

$$\underline{\lim}_{n\to\infty,\epsilon\to 0}\epsilon\log\mathbb{P}_\epsilon\left(\mathbf{N}\left(u^{\epsilon,n}(T)\right) < 4(1-\gamma)\right) \geq -\frac{2(2-\gamma-2\sqrt{1-\gamma})(12+\pi^2)}{9T}.$$

Les bornes que nous obtenons sont du même ordre en T. Elles sont à chaque fois en $\frac{1}{T}$ ce qui correspond à ce qui est obtenu en physique. Dans le cas d'une donnée initiale nulle, elles sont aussi du même ordre en γ. Nous obtenons que les queues de la masse sont, sur une échelle logarithmique, les mêmes que celles d'une loi exponentielle. Il s'agit d'un résultat également obtenu par les physiciens, voir le résultat de [63] sur l'amplitude. Dans le cas d'une donnée initiale profil de soliton, les bornes ne sont plus du même ordre en γ. Mais la borne inférieure nous permet en tout cas de conclure que les queues ne sont pas Gaussiennes. Les supposer Gaussiennes entraîne une évaluation trompeuse de l'erreur.

2.2 Grandes déviations uniformes pour des équations de Schrödinger non linéaires stochastiques avec bruit multiplicatif

Nous décrivons dans ce paragraphe un principe de grandes déviations uniforme au niveau des trajectoires des solutions d'une équation de type NLS perturbée par un bruit multiplicatif. Nous donnons des applications à l'asymptotique des temps d'explosion. Les détails ainsi que les preuves sont données dans l'annexe B qui correspond à l'article [82] publié dans *Stochastic Processes and their Applications.*

Dans cet article nous considérons des équations de Schrödinger non linéaires stochastiques avec bruit multiplicatif

$$i\mathrm{d}u^{\epsilon,u_0} - (\Delta u^{\epsilon,u_0} + \lambda|u^{\epsilon,u_0}|^{2\sigma}u^{\epsilon,u_0})\mathrm{d}t = \sqrt{\epsilon}u^{\epsilon,u_0} \circ \mathrm{d}W, \quad \lambda = \pm 1,\ x \in \mathbb{R}^d.$$

L'exposant σ satisfait $\sigma > 0$ si $d = 1, 2$ et $\sigma < \frac{2}{d-2}$ sinon et la donnée initiale u_0 est dans H^1. Nous posons $W = \Phi W_c$, où Φ est Hilbert-Schmidt de L^2 dans $\mathrm{H}^s_{\mathbb{R}}$, espace de fonctions à valeurs réelles avec $s > \frac{d}{2} + 1$ et W_c est un processus de Wiener cylindrique sur L^2. Le symbole $\circ$ correspond au produit Stratonovich. Le paramètre ϵ, intensité du bruit, tend vers zéro.

Nous prouvons dans cet article un PGD trajectoriel uniforme (en des données initiales dans des compacts de H^1). Il est lui aussi énoncé dans un espace de trajectoires explosives muni d'une topologie analogue à une topologie limite projective rendant compte des propriétés d'intégrabilité du groupe de Schrödinger. Nous notons à nouveau cet espace $\mathcal{E}_\infty$. Il est cette fois défini pour d dans $\mathbb{N}^*$ par l'ensemble des fonctions f de $\mathcal{E}(\mathrm{H}^1)$ tel que pour tout p dans $\mathcal{A}(d)$ et tout T dans $[0, \mathcal{T}(f))$, f appartienne à $\mathrm{L}^{r(p)}\left(0, T; \mathrm{W}^{1,p}\right)$.

L'ensemble d'exposants $\mathcal{A}(d)$ est l'ensemble $[2,\infty)$ lorsque $d=1$ ou $d=2$ et respectivement $\left[2,\frac{2(3d-1)}{3(d-1)}\right)$ et $\left[2,\frac{2d}{d-1}\right)$ lorsque $d\geq 3$. L'espace $\mathcal{E}_\infty$ est muni de la topologie définie pour φ_1 dans $\mathcal{E}_\infty$ par la base de voisinages

$$W_{T,p,r}(\varphi_1)=\{\varphi\in\mathcal{E}_\infty:\mathcal{T}(\varphi)>T,\ \|\varphi_1-\varphi\|_{X^{(T,p)}}\leq r\}$$

où $T<\mathcal{T}(\varphi_1)$, p appartient à $\mathcal{A}(d)$ et r est strictement positif. On peut vérifier que cet espace topologique est bien séparé. Si on note toujours par $\mathcal{T}:\mathcal{E}_\infty\to[0,\infty]$ le temps d'explosion, l'application $\mathcal{T}$ est mesurable et semi-continue inférieurement.

Le squelette de l'équation stochastique est l'unique solution mild du problème de contrôle

$$\left\{\begin{array}{l} i\frac{\mathrm{d}u}{\mathrm{d}t}=\Delta u+\lambda|u|^{2\sigma}u+u\Phi h,\\ u(0)=u_0\ \in \mathrm{H}^1,\ h\in \mathrm{L}^2\left(0,\infty;\mathrm{L}^2\right).\end{array}\right.$$

Le résultat est énoncé ci-après, nous notons $K\subset\subset \mathrm{H}^1$ lorsque K est un compact de H^1, $Int(A)$ l'intérieur de A et $\mathcal{B}(\mathcal{E}_\infty)$ les boréliens de $\mathcal{E}_\infty$.

Theorem 2.2.1 *La famille des lois des trajectoires $\left(\mu^{u^{\epsilon,u_0}}\right)_{\epsilon>0}$ satisfait un PGD uniforme de vitesse ϵ et de bonne fonction de taux*

$$I^{u_0}(w)=\tfrac{1}{2}\inf_{h\in\mathrm{L}^2(0,\infty;\mathrm{L}^2):w=\mathbf{S}^c(u_0,h)}\|h\|^2_{\mathrm{L}^2(0,\infty;\mathrm{L}^2)},$$

i.e. quelque soit $K\subset\subset\mathrm{H}^1$, et A dans $\mathcal{B}(\mathcal{E}_\infty)$, nous avons la borne inférieure

$$-\sup_{u_0\in K}\inf_{w\in Int(A)}I^{u_0}(w)\leq \underline{\lim}_{\epsilon\to 0}\epsilon\log\inf_{u_0\in K}\mathbb{P}\left(u^{\epsilon,u_0}\in A\right)$$

et la borne supérieure

$$\overline{\lim}_{\epsilon\to 0}\epsilon\log\sup_{u_0\in K}\mathbb{P}\left(u^{\epsilon,u_0}\in A\right)\leq-\inf_{w\in\overline{A},u_0\in K}I^{u_0}(w).$$

Ce résultat se déduit cette fois d'un PGD pour le processus de Wiener (et non la convolution stochastique). La preuve est ici plus complexe que dans le cas d'un bruit additif. En effet, les trajectoires des solutions ne sont pas images directes par une application continue des trajectoires du processus de Wiener.

Il serait aussi envisageable d'utiliser un principe de contraction et de prouver un PGD pour les trajectoires rugueuses au dessus des solutions à partir d'un PGD pour les trajectoires rugueuses au dessus du processus de Wiener. Cela nécessiterait par contre de montrer un résultat de continuité à la T. Lyons, *c.f.* [107, 108], mais pour l'EDP stochastique. Nous pourrions notamment nous inspirer de [105, 103].

Nous adoptons ici une preuve plus classique basée essentiellement sur le lemme d'Azencott aussi appelé inégalité de Freidlin-Wentzell. Ce lemme correspond à ce qui remplace ici la continuité au niveau des trajectoires. Il s'énonce de la manière suivante, C_a désigne le ensembles des niveaux inférieurs à a de la fonction de taux d'un PGD pour le processus de Wiener. Le squelette $\mathbf{S}(u_0, f)$ est cette fois ci celui où l'on remplace Φh par $\frac{\partial f}{\partial t}$.

Proposition 2.2.2 *Pour tout a, R et ρ strictement positifs, u_0 dans H^1, f dans C_a, $T < \mathcal{T}(\mathbf{S}(u_0, f))$, p dans $\mathcal{A}(d)$, il existe ϵ_0, γ et r strictement positifs tels que pout tout ϵ dans $(0, \epsilon_0]$ et $\tilde{u}_0$ dans $B_{\mathrm{H}^1}(u_0, r)$,*

$$\epsilon \log \mathbb{P}\left(\left\| u^{\epsilon, \tilde{u}_0} - \mathbf{S}(u_0, f) \right\|_{X^{(T,p)}} \geq \rho ; \left\| \sqrt{\epsilon} W - f \right\|_{\mathrm{C}\left([0,T]; \mathrm{H}^s_{\mathbb{R}}\right)} < \gamma \right) \leq -R.$$

La preuve de ce lemme nécessite les estimées de décroissance exponentielle des queues dans des espaces de Banach suivantes.

Proposition 2.2.3 *Si Z, défini par $Z(t) = \int_0^t U(t-s)\xi(s) dW(s)$, est tel qu'il existe η positif tel que $\|\xi\|^2_{\mathrm{C}([0,T];\mathrm{H}^1)} \leq \eta$ p.s., alors pour tout p dans $\tilde{\mathcal{A}}(d)$, T et δ positifs,*

$$\mathbb{P}\left(\|Z\|_{\mathrm{C}([0,T];\mathrm{H}^1)} \geq \delta \right) \leq 3 \exp\left(-\frac{\delta^2}{\kappa_1(\eta)} \right)$$
$$\mathbb{P}\left(\|Z\|_{\mathrm{L}^{r(p)}(0,T;W^{1,p})} \geq \delta \right) \leq c \exp\left(-\frac{\delta^2}{\kappa_2(\eta)} \right)$$

où $c = 2e + \exp\left((2ek_0!)^{\frac{1}{k_0}} \right)$, $k_0 = \max(2, \min\{k \in \mathbb{N} : 2k \geq r(p)\})$

$$\kappa_1(\eta) = T 4 c(\infty)^2 \|\Phi\|^2_{\mathcal{L}_2^{0,s}} \eta,$$

$$\kappa_2(\eta) = \frac{8c\left(\frac{r(p)d}{2}\right)^2 T^{1-\frac{2}{r(p)}} (d+1)(d+p) \|\Phi\|^2_{\mathcal{L}_2^{0,s}}}{1 - \frac{4}{r(p)}} \eta,$$

$c\left(\frac{r(p)d}{2}\right)$ *et* $c(\infty)$ *sont les normes des injections continues* $\mathrm{H}^s_{\mathbb{R}} \subset \mathrm{W}^{1,\frac{r(p)d}{2}}_{\mathbb{R}}$ *et* $\mathrm{H}^s_{\mathbb{R}} \subset \mathrm{W}^{1,\infty}_{\mathbb{R}}$.

Nous utilisons également la continuité du squelette modifié par rapport aux contrôles sur les ensembles C_a des niveaux inférieurs à a strictement positifs. Le résultat de continuité s'énonce comme suit.

Proposition 2.2.4 *Pour tout u_0 dans H^1, a positif et f dans C_a, $\mathbf{S}(u_0, f)$ existe et est défini de manière unique. L'application est continue de $\mathrm{H}^1 \times C_a$ dans $\mathcal{E}_\infty$, où C_a a la topologie induite par celle de $\mathrm{C}\left([0,\infty); \mathrm{H}^s_{\mathbb{R}}\right)$.*

Nous donnons dans cet article une première application du PGD. Il s'agit d'une application aux temps d'explosion. Si on note $u_d^{u_0}$ la solution de l'équation déterministe, nous prouvons

Proposition 2.2.5 *Si $T < \inf_{u_0 \in K} \mathcal{T}\left(u_d^{u_0}\right)$, où $K \subset\subset \mathrm{H}^1$, il existe c strictement positif tel que*

$$\overline{\lim}_{\epsilon \to 0} \epsilon \log \sup_{u_0 \in K} \mathbb{P}\left(\mathcal{T}(u^{\epsilon, u_0}) \leq T\right) \leq -c.$$

Proposition 2.2.6 *Soit U^{u_0} la solution de l'équation de Schrödinger libre avec une donnée initiale u_0 dans H^s et supposons que l'espace vectoriel engendré par $\left\{|U^{u_0}(t)|^2, t \in [0, 2T]\right\}$ appartienne à l'image de Φ pour $T > \mathcal{T}\left(u_d^{u_0}\right)$. Il existe alors c strictement positif tel que*

$$\underline{\lim}_{\epsilon \to 0} \epsilon \log \mathbb{P}\left(\mathcal{T}(u^{\epsilon, u_0}) > T\right) \geq -c.$$

2.3 Asymptotique de petits bruits pour la fluctuation des temps d'arrivée dans la transmission par solitons

Nous décrivons dans ce paragraphe une application des principes de grandes déviations trajectoriels pour les solutions d'équations de type NLS perturbée par un bruit additif ou multiplicatif. Nous étudions l'asymptotique des queues du temps d'arrivée d'une donnée initiale profil de soliton lorsque l'amplitude ϵ du bruit tend vers 0. L'article [44] correspondant figure en annexe C. La fluctuation du temps d'arrivée est une source d'erreur de transmission. Nous avons commencé à traiter le sujet des erreurs de transmission dans la section 2.1. Nous considérons toujours le cas de l'équation avec non-linéarité cubique focalisante en dimension 1.

Nous étudions dans un premier temps le cas d'un petit bruit complexe additif. Il correspond à l'émission spontanée de bruit par des amplificateurs régulièrement espacés le long de la fibre afin de palier à la perte par amortissement. Nous considérons aussi le cas de petits bruits multiplicatifs réels. Ceux ci correspondent à d'autres types d'amplification : l'amplification de

Raman et l'amplification utilisant le mélange de quatre ondes. La donnée initiale peut être ou bien nulle ou un profil de soliton

$$\Psi_A^0(x) = \sqrt{2}A\text{sech}(Ax).$$

Ceci correspond à l'émission d'un 1 ou d'un 0.

Dans le cas du bruit additif les fluctuations de la masse

$$\mathbf{N}\left(u^{\epsilon,\Psi^0_A}(T)\right) = \left\|u^{\epsilon,\Psi^0_A}(T)\right\|^2_{\mathrm{L}^2},$$

et du temps d'arrivée (la position)

$$\mathbf{Y}\left(u^{\epsilon,u_0}(T)\right) = \int_{\mathbb{R}} x \left|u^{\epsilon,\Psi^0_A}(T,x)\right|^2 dx$$

en l'extrémité T de la fibre sont supposées être les facteurs les plus limitant dans la transmission par solitons. Nous rappelons les résultats obtenus dans [81] sur les queues de la masse en l'extrémité de la fibre et nous menons une étude similaire pour la position cette fois. Notons que la fluctuation de la position n'est pénalisante que pour des données initiales $\Psi_A^0(x)$. Nous n'étudierons donc que ce cas là. Par ailleurs nous traitons cette fois le cas d'un bruit additif et d'un bruit multiplicatif. Dans le cas du bruit multiplicatif, la masse est conservée et seule la position fluctue. Nous donnons dans ce cas également, mais pour un bruit légèrement différent, des majorations et minorations des queues de la position dans l'asymptotique de petits bruits.

L'asymptotique lorsque le bruit tend vers 0 (*i.e.* ϵ tend vers 0), sur une échelle logarithmique, des queues de distribution est caractérisée par un problème de contrôle optimal pour une équation de Schrödinger non linéaire contrôlée. Nous ne résolvons pas ce problème mais donnons des bornes inférieures et supérieures et en déduisons des bornes inférieures et supérieures des queues. Pour obtenir des bornes supérieures nous cherchons des bornes inférieures du problème de contrôle optimal. Celles-ci sont obtenues par des inégalités d'énergie pour l'équation contrôlée. Pour obtenir des bornes inférieures nous cherchons un majorant le plus précis possible du problème de contrôle optimal. Pour cela nous effectuons la minimisation sur un ensemble de fonctions plus petit constitué de solitons avec un nombre fini de paramètres, les paramètres étant fonction du temps. Le majorant est alors donné par un problème de calcul des variations. Celui-ci nous permet de deviner un candidat qui nous permettra d'obtenir une borne. Nous obtenons des bornes supérieures et inférieures du même ordre de grandeur en T

(T est très grand) longueur de la fibre et en R (de l'ordre de l'unité). L'ordre de grandeur en l'amplitude A de la donnée initiale est celui obtenu par les physiciens pour le bruit additif.

Pour l'étude de la fluctuation de la position nous avons besoin de PGDs légèrement différents des PGDs donnés précédemment. En effet, nous souhaitons nous placer dans un espace de trajectoires où la position est bien définie. Nous introduisons donc les espaces

$$\Sigma = \left\{ f \in \mathrm{H}^1 : \ x \mapsto x f(x) \in \mathrm{L}^2 \right\},$$

et

$$\Sigma^{\frac{1}{2}} = \left\{ f \in \mathrm{H}^1 : \ x \mapsto \sqrt{|x|} f(x) \in \mathrm{L}^2 \right\}$$

munis des normes

$$\|f\|_{\Sigma}^2 = \|f\|_{\mathrm{H}^1}^2 + \|x \mapsto x f(x)\|_{\mathrm{L}^2}^2,$$

$$\|f\|_{\Sigma^{\frac{1}{2}}}^2 = \|f\|_{\mathrm{H}^1}^2 + \left\| x \mapsto \sqrt{|x|} f(x) \right\|_{\mathrm{L}^2}^2.$$

Nous prouvons alors

Théorème 2.3.1 *Supposons que Φ (tel que $W = \Phi W_c$ soit le processus de Wiener dirigeant l'équation stochastique) soit Hilbert-Schmidt de L^2 dans Σ dans le cas additif et de L^2 dans $\mathrm{H}^s(\mathbb{R}, \mathbb{R})$) avec $s > 3/2$ dans le cas multiplicatif. Supposons que la donnée initiale u_0 appartienne à Σ. Alors les solutions des équations stochastiques sont presque sûrement dans $\mathrm{C}([0,T]; \Sigma^{\frac{1}{2}})$. De plus elles définissent de variables aléatoires à valeurs dans $\mathrm{C}([0,T]; \Sigma^{\frac{1}{2}})$ et leurs lois $\left(\mu^{u^{\epsilon, u_0}}\right)_{\epsilon > 0}$ satisfont des PGDs de vitesse ϵ et de bonnes fonctions de taux*

$$I^{u_0}(w) = \frac{1}{2} \inf_{h \in \mathrm{L}^2(0,T;\mathrm{L}^2): \ w = \mathbf{S}(u_0, h)} \|h\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}^2,$$

où $\mathbf{S}(u_0, \cdot) = \mathbf{S}^{a,u_0}(\cdot)$ dans le cas additif et $\mathbf{S}(u_0, \cdot) = \mathbf{S}^{m,u_0}(\cdot)$ dans le cas multiplicatif, avec la convention $\inf \emptyset = \infty$.

Le squelette $\mathbf{S}^{a,u_0}$ de l'énoncé ci-dessus est la solution mild du problème de contrôle

$$\begin{cases} i \frac{\mathrm{d}u}{\mathrm{d}t} = \Delta u + |u|^2 u + \Phi h, \\ u(0) = u_0 \ \in \Sigma \text{ and } h \in \mathrm{L}^2\left(0, T; \mathrm{L}^2\right). \end{cases}$$

Le squelette $\mathbf{S}^{m,u_0}$ est la solution mild du problème de contrôle où l'équation est remplacée par

$$i \frac{\mathrm{d}u}{\mathrm{d}t} = \Delta u + |u|^2 u + u \Phi h.$$

La borne supérieure qui suit s'obtient en considérant la relation

$$\begin{aligned}\mathbf{Y}(\mathbf{S}^{a,\Psi_A^0}(h)(t)) = \quad & 4\mathfrak{Re}\left(\int_0^t\int_0^s\int_{\mathbb{R}}\overline{\mathbf{S}^{a,\Psi_A^0}(h)(\sigma,x)}\left(\partial_x\Phi h\right)(\sigma,x)dxd\sigma ds\right)\\ & -2\mathfrak{Re}\left(i\int_0^t\int_{\mathbb{R}}x\overline{\mathbf{S}^{a,\Psi_A^0}(h)(s,x)}\left(\Phi h\right)(s,x)dxds\right).\end{aligned}$$

Proposition 2.3.1 *Quels que soient T, A et R positifs et Φ opérateur Hilbert-Schmidt de L^2 dans Σ, nous avons l'inégalité suivante*

$$\overline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(\mathbf{Y}\left(u^{\epsilon,\Psi_A^0}(T)\right)\geq R\right)\leq -\frac{R^2}{8T(2T+1)^2\left(4A+\frac{R}{2T+1}\right)\|\Phi\|^2_{\mathcal{L}_c(\mathrm{L}^2,\Sigma)}}.$$

En ce qui concerne la borne inférieure dans le cas du bruit additif, l'approche par le calcul des variations nous permet de deviner la forme des contrôles la plus adaptée afin d'obtenir une borne inférieure en $\frac{1}{T^3}$ correspondant à la physique et à la borne supérieure lorsque T est grand (R quand à lui est de l'ordre de l'unité, cela donne malgré tout un évènement de grandes déviations car la variance est multipliée par ϵ qui tend vers 0). Puis en partant de l'équation contrôlée elle même et grâce à des transformations usuelles (transformation de Gauge, méthode des caractéristiques...) nous avons pu déduire la forme de soliton modulé présentée ci-après.

Nous considérons la "limite bruit blanc" et définissons l'ensemble de contrôles admissibles pour A et T positifs et D sous ensemble de $[R,R+1]$ pour R positif fixé

$$\begin{aligned}\mathcal{H}^D_{A,T} = \quad & \{h\in\mathrm{L}^2(0,T;\mathrm{L}^2),\ h(t,x)=\lambda(t)\left(x-2\int_0^t\int_0^s\lambda(\tau)d\tau ds\right)\tilde{\Psi}_{A,\lambda}(t,x),\\ & \qquad\qquad\text{avec }\lambda(t)=\tfrac{3\tilde{R}(T-t)}{8AT^3},\ \tilde{R}\in D\}\end{aligned}$$

où

$$\begin{aligned}&\tilde{\Psi}_{A,\lambda}(t,x)=\sqrt{2}A\mathrm{sech}\left(A\left(x-2\int_0^t\int_0^s\lambda(\tau)d\tau ds\right)\right)\exp\left(2i\int_0^t\lambda(s)\int_0^s\int_0^\tau\lambda(\sigma)d\sigma d\tau ds\right)\\ &\exp\left[-iA^2t+i\int_0^t\left(\int_0^s\lambda(\tau)d\tau\right)^2ds-ix\int_0^t\lambda(s)ds+2i\left(\int_0^t\lambda(s)ds\right)\left(\int_0^t\int_0^s\lambda(\tau)d\tau ds\right)\right].\end{aligned}$$

Remarquons qu'il suffit alors d'optimiser en la fonction λ de $\mathrm{L}^1(0,T)$. Les conditions aux limites ne sont pas des conditions usuelles et un calcul formel nous permet de deviner l'élément $\lambda(t)=\frac{3\tilde{R}(T-t)}{8AT^3}$ qui pourrait être optimal. Nous obtenons la borne qui suit.

Proposition 2.3.2 *Quels que soient T, A et R positifs. Supposons que pour D dense dans $[R,R+1]$, $(\Phi_n)_{n\in\mathbb{N}}$ soit une suite d'opérateurs de L^2 dans Σ*

telle que pour tout h dans $\mathcal{H}^D_{T,A}$, $\Phi_n h$ converge vers h dans $\mathrm{L}^1(0,T;\Sigma)$. Alors l'inégalité suivante est vérifiée, l'exposant n rappelle que Φ est remplacé par Φ_n,

$$\underline{\lim}_{n\to\infty,\epsilon\to 0}\epsilon\log\mathbb{P}\left(\mathbf{Y}\left(u^{\epsilon,\Psi^0_A,n}(T)\right)\geq R\right)\geq -\frac{\pi^2R^2}{128T^3A^3}.$$

Les deux bornes sont en $-\frac{1}{T^3}$ lorsque T est grand. Cela confirme que la fluctuation du centre est prépondérante sur la fluctuation de la masse en ce qui concerne l'erreur de transmission. En effet, T étant très grand, les queues de la position sont plus épaisses que celles du centre. En outre sur une échelle logarithmique les queues sont indistingables de queues Gaussiennes. Si la loi était effectivement Gaussienne, le facteur T^3 correspondrait exactement à la variance proportionnelle à T^3 de l'effet Gordon-Haus. Cette variance en T^3 est supérieure à celle du mouvement Brownien en T et on appelle aussi cette fluctuation une super diffusion. Une partie de la littérature physique est consacrée à la loi de la position. Nous retrouvons ici le fait qu'au premier ordre on peut bien considérer que la loi de la position est Gaussienne. Si nous considérons également la suite d'opérateurs $(\Phi_n)_{n\in\mathbb{N}}$ dans la borne supérieure, le comportement en A grand n'est pas contradictoire. Nous obtenons que l'ordre de grandeur en A du logarithme de la queue est supérieur à $-\frac{1}{A^3}$. Il s'agit aussi de l'ordre de grandeur obtenu en physique.

Dans le cas des bruits multiplicatifs nous ne sommes pas arrivés à obtenir une borne inférieure car les contrôles suggérés par l'approche calcul des variations ne sont ni dans L^2 ni dans l'image de Φ ($\mathrm{H}^s(\mathbb{R},\mathbb{R})$, $s>\frac{3}{2}$). Nous obtenons par contre une borne supérieure. Nous considérons ensuite que le bruit est à valeurs dans $\mathrm{H}^s(\mathbb{R},\mathbb{R})\oplus x\mathrm{L}^1(0,T;\mathbb{R})$. Alors, après avoir donné un sens aux équations stochastiques et au squelette associé nous prouvons la borne inférieure suivante.

Proposition 2.3.3 *Quels que soient T, A et R positifs, l'inégalité suivante est satisfaite*

$$\underline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(\mathbf{Y}\left(u^{\epsilon,\Psi^0_A}(T)\right)\geq R\right)\geq -\frac{3R^2}{128A^2T^3}$$

Pour cette borne nous exploitons l'identité d'énergie qui suit.

$$\mathbf{Y}\left(\mathbf{S}^{m,\Psi^0_A}(h)(t)\right)=2\mathfrak{Re}\left(i\int_0^t\int_{\mathbb{R}}\overline{\mathbf{S}^{m,\Psi^0_A}(h)(s,x)}\partial_x\mathbf{S}^{m,\Psi^0_A}(h)(s,x)dxds\right),$$

Nous obtenons également la borne supérieure correspondante.

Proposition 2.3.4 *Quels que soient T, A et R positifs, l'inégalité suivante est satisfaite*

$$\overline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(\mathbf{Y}\left(u^{\epsilon,\Psi_A^0}(T)\right) \geq R\right) \leq -\left(\frac{3}{16}\right)^2 \frac{R^2}{A^2T^3\left(\|\Phi\|^2_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{W}^{1,\infty}(\mathbb{R},\mathbb{R}))} \vee 1\right)}.$$

Nous obtenons, comme dans le cas additif, le facteur $-\frac{1}{T^3}$ et que sur une échelle logarithmique les queues sont bien indistingables des queues Gaussiennes. Il s'agit bien du résultat auquel on s'attend d'après la référence [59]. Les ordres de grandeur en A sont en $-\frac{1}{A^2}$. Nous nous attendrions pourtant dans ce cas, au vu de [59] à un ordre de grandeur en $-\frac{1}{A^4}$.

2.4 Application à la sortie d'un domaine d'attraction pour des équations de Schrödinger non linéaires stochastiques faiblement amorties

Nous présentons dans cette section une autre application des principes de grandes déviations trajectoriels pour les solutions d'équations de type NLS faiblement amorties perturbée par un bruit additif ou multiplicatif. Nous étudions ici l'asymptotique lorsque l'amplitude du bruit tend vers 0 du temps moyen et du point de sortie d'un voisinage de 0. L'article [84] correspondant figure en annexe D.

Dans cet article nous étudions des équations faiblement amorties. Dans le cas du bruit additif nous avons

$$i\mathrm{d}u^{\epsilon,u_0} = (\Delta u^{\epsilon,u_0} + \lambda|u^{\epsilon,u_0}|^{2\sigma}u^{\epsilon,u_0} - i\alpha u^{\epsilon,u_0})\mathrm{d}t + \sqrt{\epsilon}\mathrm{d}W, \quad \lambda = \pm 1,\ x \in \mathbb{R}^d$$

où α et ϵ sont strictement positifs et où la donnée initiale u_0 est dans L^2 ou H^1. Le bruit est toujours coloré en espace et le processus de Wiener est à valeur L^2 ou H^1. Quand le bruit est multiplicatif réel avec produit Stratonovich, l'équation s'écrit

$$i\mathrm{d}u^{\epsilon,u_0} = (\Delta u^{\epsilon,u_0} + \lambda|u^{\epsilon,u_0}|^{2\sigma}u^{\epsilon,u_0} - i\alpha u^{\epsilon,u_0})\mathrm{d}t + \sqrt{\epsilon}u^{\epsilon,u_0}\circ\mathrm{d}W, \quad \lambda = \pm 1,\ x \in \mathbb{R}^d.$$

Le processus de Wiener est alors à valeurs dans $\mathrm{H}^s_{\mathbb{R}}$ avec $s > \frac{d}{2} + 1$, espace de Sobolev basé sur L^2 de fonctions à valeurs réelles. Les données initiales sont dans ce cas dans H^1. Dans L^2 le phénomène de sortie n'existe pas car la masse décroît. On peut trouver des résultats sur les équations déterministes par exemple dans [86, 90, 138].

En l'absence de bruit les solutions tendent vers zéro dans L^2 (respectivement dans H^1). En présence de bruit par contre, les trajectoires aléatoires sortent de voisinages de zéro invariants par le flot de l'équation déterministe. Le comportement est donc en cela complètement différent de celui du système déterministe. Le temps de sortie est exponentiel et plus le bruit est faible plus le temps est grand. Mais aussi, les trajectoires qui sortent du domaine ou les points de sortie les plus probables minimisent une certaine énergie (énergie "effective" en physique).

En présence d'un unique extremum le comportement le plus probable est le comportement qui minimise l'énergie et l'évolution est essentiellement déterministe. Sinon, à chaque trajectoire ou point de sortie est associé un poids. Notons que l'étude menée dans [67] correspond au cas où plusieurs équilibres coexistent.

L'étude de ce problème trouve par exemple des applications en physique (mécanique quantique et statistique, optique...). Elle intervient également en économie et en particulier en macro économie financière. Voici donc un exemple d'application à l'optique ou l'hydrodynamique et plus généralement tous les domaines où intervient notre EDPS.

Nous menons dans ce papier une étude analogue à celle initiée par Freidlin et Wentzell pour les équations différentielles perturbées par un petit bruit additif, *c.f.* [73] Chapitre 4. Cette étude a été généralisée au cas de bruits multiplicatifs mais la dimension finie est utilisée très régulièrement dans la preuve afin d'obtenir de la compacité *c.f.* par exemple [48] Chapitre 5. Le cas de la dimension infinie est plus complexe, les propriétés de compacité disparaissent en général et le semi-groupe d'évolution n'est pas uniformément continu mais seulement fortement continu. Ce problème a déjà été étudié pour certaines EDPS, *c.f.* par exemple [29, 34, 68]. Le problème que nous étudions pose certaines difficultés particulières : le groupe n'a pas de propriétés régularisantes, les variables d'espace vivent dans tout l'espace $\mathbb{R}^d$, la non linéarité n'est jamais localement Lipschitzienne sauf lorsque $d = 1$ pour des solutions dans H^1.

Nous montrons les deux théorèmes qui suivent valables dans L^2 (pour un bruit additif) et dans H^1 (pour un bruit additif ou multiplicatif). Le domaine D est un borélien de L^2 (respectivement H^1) borné dans L^2 (respectivement H^1) contenant zéro dans son intérieur et invariant par le flot de l'équation déterministe. Le temps de sortie est défini par

$$\tau^{\epsilon,u_0} = \inf\left\{t \geq 0 : \ u^{\epsilon,u_0}(t) \in D^c\right\}.$$

Nous utiliserons pour l'étude de ce problème des PGDs et notons donc $\mathbf{S}(u_0, h)$ le squelette de l'équation stochastique, *i.e.* la solution mild de l'équation contrôlée

$$\begin{cases} i\left(\frac{\mathrm{d}u}{\mathrm{d}t} + \alpha u\right) = \Delta u + \lambda |u|^{2\sigma} u + \Phi h, \\ u(0) = u_0 \end{cases}$$

où u_0 appartient à L^2 ou H^1 dans le cas additif ou

$$\begin{cases} i\left(\frac{\mathrm{d}u}{\mathrm{d}t} + \alpha u\right) = \Delta u + \lambda |u|^{2\sigma} u + u\Phi h, \\ u(0) = u_0 \end{cases}$$

où u_0 appartient à H^1 dans le cas multiplicatif.

La fonction de taux des PGDs trajectoriels est alors toujours définie comme

$$I_T^{u_0}(w) = \frac{1}{2} \inf_{h \in \mathrm{L}^2(0,T;\mathrm{L}^2):\ \mathbf{S}(u_0,h)=w} \int_0^T \|h(s)\|_{\mathrm{L}^2}^2 ds.$$

Nous notons dans ce qui suit par $\mathcal{N}(B,\rho)$ pour ρ positif le $\rho-$voisinage dans L^2 de l'ensemble B.

Nous définissons

$$\overline{e} = \inf\left\{ I_T^0(w) :\ w(T) \in \overline{D}^c,\ T > 0 \right\}.$$

Si ρ est positif et suffisamment petit nous posons

$$e_\rho = \inf\left\{ I_T^{u_0}(w) : \|u_0\|_{\mathrm{L}^2} \le \rho,\ w(T) \in (D_{-\rho})^c,\ T > 0 \right\},$$

où $D_{-\rho} = D \setminus \mathcal{N}(\partial D, \rho)$ et ∂D désigne le bord de D dans L^2.

Enfin nous définissons

$$\underline{e} = \lim_{\rho \to 0} e_\rho.$$

Dans le cas H^1, il convient de remplacer dans ce qui précède L^2 par H^1.

Dans tous les cas $0 < \underline{e} \le \overline{e}$. Nous ne montrons pas dans ce papier que $\underline{e} = \overline{e}$, c'est un problème de contrôle qui semble difficile et qui sera étudié ultérieurement.

Le résultat sur le temps de sortie s'énonce dans L^2 et dans H^1 pour des domaines dans L^2 ou dans H^1 comme suit.

Theorem 2.4.1 *Quel que soit u_0 dans D et δ strictement positif, il existe L strictement positif tel que*

$$\overline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(\tau^{\epsilon,u_0} \notin \left(\exp\left(\frac{\underline{e}-\delta}{\epsilon}\right), \exp\left(\frac{\overline{e}+\delta}{\epsilon}\right)\right)\right) \leq -L,$$

et quelque soit u_0 dans D,

$$\underline{e} \leq \underline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{E}\left(\tau^{\epsilon,u_0}\right) \leq \overline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{E}\left(\tau^{\epsilon,u_0}\right) \leq \overline{e}.$$

De plus, quel que soit δ strictement positif, il existe L strictement positif tel que

$$\overline{\lim}_{\epsilon\to 0}\epsilon \log \sup_{u_0\in D} \mathbb{P}\left(\tau^{\epsilon,u_0} \geq \exp\left(\frac{\overline{e}+\delta}{\epsilon}\right)\right) \leq -L,$$

et

$$\overline{\lim}_{\epsilon\to 0}\epsilon \log \sup_{u_0\in D} \mathbb{E}\left(\tau^{\epsilon,u_0}\right) \leq \overline{e}.$$

Le deuxième théorème caractérise formellement les points de sortie. Nous définissons pour ρ strictement positif et suffisamment petit, N un fermé de ∂D (pour la topologie de L^2, respectivement H^1) dans L^2 (respectivement H^1),

$$e_{N,\rho} = \inf\left\{I_T^{u_0}(w) : \|u_0\|_{\mathrm{L}^2} \leq \rho,\ w(T) \in \left(D \setminus \mathcal{N}(N,\rho)\right)^c,\ T > 0\right\},$$

dans le cas H^1 remplacer $\|u_0\|_{\mathrm{L}^2}$ par $\|u_0\|_{\mathrm{H}^1}$ et $\mathcal{N}(N,\rho)$ par le $\rho-$voisinage dans H^1. Nous posons ensuite

$$\underline{e}_N = \lim_{\rho\to 0} e_{N,\rho}.$$

Comme $e_\rho \leq e_{N,\rho}$ nous avons bien $\underline{e} \leq \underline{e}_N$. Nous prouvons donc

Théorème 2.4.1 *Si $\underline{e}_N > \overline{e}$, alors quel que soit u_0 dans D, il existe L strictement positif tel que*

$$\overline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(u^{\epsilon,u_0}\left(\tau^{\epsilon,u_0}\right) \in N\right) \leq -L.$$

Nous pouvons en déduire le corollaire

Corollaire 2.4.2 *Supposons que v^* appartenant à ∂D soit tel que quel que soit δ strictement positif et $N = \{v \in \partial D :\ \|v - v^*\|_{\mathrm{L}^2} \geq \delta\}$ (dans le cas H^1 noter $\|v - v^*\|_{\mathrm{H}^1}$) nous avons $\underline{e}_N > \overline{e}$ alors*

$$\forall \delta > 0,\ \forall u_0 \in D,\ \exists L > 0 :\ \overline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(\left\|u^{\epsilon,u_0}\left(\tau^{\epsilon,u_0}\right) - v^*\right\|_{\mathrm{L}^2} \geq \delta\right) \leq -L,$$

(dans le cas H^1 noter $\|\cdot\|_{\mathrm{H}^1}$).

Ces résultats sont démontrés grâce à des PGDs uniformes en les données initiales avec une formulation des bornes supérieures et inférieures du type Freidlin-Wentzell. Nous exploitons en particulier les propriétés d'intégrabilité du groupe libre de Schrödinger, l'évolution de la masse et de l'Hamiltonien, des estimées exponentielles de queues d'intégrales stochastiques et la propriété de Markov de la solution. Nous verrons que le cas d'un bruit multiplicatif se traite comme celui d'un bruit additif, l'argument de troncature est seulement légèrement différent.

Nous concluons en remarquant que si nous étions capables de prouver $\underline{e} = \overline{e}$ pour la sortie d'un domaine de H^1 avec un bruit additif blanc en espace et en temps, la caractérisation du point de sortie serait reliée aux ondes solitaires. Remarquons qu'il a été obtenu numériquement dans [46] pour des équations de Korteweg-de Vries que l'énergie injectée par le bruit organise le système et crée des ondes solitaires. Cela serait peut être une confirmation de ce fait.

2.5 Grandes déviations et théorèmes de support, le cas d'une équation en dimension 1 avec bruit fractionnaire additif

Dans cette section nous nous intéressons à des équations de Schrödinger non linéaires stochastiques avec un bruit additif fractionnaire en dimension 1. Il s'agit donc d'une extension des résultats de l'article [81] à des bruits Gaussiens plus généraux et colorés en temps. Les preuves sont données dans l'article qui figure en annexe E.

L'équation stochastique s'écrit

$$i\mathrm{d}u - (\Delta u + \lambda |u|^{2\sigma} u)\mathrm{d}t = \mathrm{d}W^H, \quad \lambda = \pm 1,$$

la donnée initiale u_0 est une fonction de H^1. Nous considérons à nouveau des solutions faibles au sens de l'analyse des équations aux dérivées partielles, ou de manière équivalente à des solutions milds

$$u(t) = U(t)u_0 - i\lambda \int_0^t U(t-s)(|u(s)|^{2\sigma} u(s))ds - i\int_0^t U(t-s)dW^H(s),$$

où $(U(t))_{t\in\mathbb{R}}$ est le groupe de Schrödinger sur H^1. Le processus W^H est un processus de Wiener fractionnaire. Le paramètre H, paramètre de Hurst, appartient à $(0,1)$. Nous supposons que W^H est l'image directe par Φ

Hilbert-Schmidt d'un processus de Wiener cylindrique fractionnaire sur L^2. Le processus cylindrique est tel que pour toute base Hilbertienne $(e_j)_{j\in\mathbb{N}}$ de L^2, il existe des mouvements Browniens fractionnaires $\left(\beta_j^H(t)\right)_{t\geq 0}$ tels que $W_c(t)=\sum_{j\in\mathbb{N}}\beta_j^H(t)e_j$. Nous faisons l'hypothèse (A) suivante

$$\begin{gathered}\Phi \text{ est Hilbert-Schmidt de } \mathrm{L}^2 \text{ dans } \mathrm{H}^{1+\gamma}\\ \text{avec } (1-2H)<\gamma<1 \text{ si } H<\tfrac{1}{2} \text{ et } 0\leq\gamma<1 \text{ si } H>\tfrac{1}{2}.\end{gathered}$$

Un mouvement Brownien fractionnaire est un processus Gaussien centré à accroissements stationnaires

$$\mathbb{E}\left(\left|\beta^H(t)-\beta^H(s)\right|^2\right)=|t-s|^{2H},\quad t,s>0,$$

de covariance

$$\mathbb{E}\left(\beta^H(t)\beta^H(s)\right)=\frac{1}{2}\left(s^{2H}+t^{2H}-|s-t|^{2H}\right).$$

La covariance des accroissements passés et futurs est négative si $H<\frac{1}{2}$ et positive si $H>\frac{1}{2}$. Le cas $H=\frac{1}{2}$ où les accroissements sont indépendants est celui du mouvement Brownien. Le processus admet une modification à trajectoires $\alpha-$Höldériennes pour $\alpha<H$. Il peut s'écrire, quitte à élargir l'espace de probabilités, sous la forme

$$\beta^H(t)=\int_0^t K^H(t,s)d\beta(s),$$

où K^H est le noyau de carré intégrable triangulaire, *i.e.* $K^H(t,s)=0$ si $s>t$,

$$K^H(t,s)=c_H(t-s)^{H-\frac{1}{2}}+c_H\left(\frac{1}{2}-H\right)\int_s^t(u-s)^{H-\frac{3}{2}}\left(1-\left(\frac{s}{u}\right)^{\frac{1}{2}-H}\right)du, \tag{2.5.1}$$

pour une certaine constante c_H. Ce n'est pas une semi-martingale et nous considérons l'intégrale stochastique comme une intégrale de Skohorod, comme cela a été défini dans [3].

Cette intégrale se réécrit comme une intégrale de Skohorod pour le mouvement Brownien. Cela revient à définir dans un premier temps un espace de Hilbert noyau auto reproduisant au niveau du bruit, noté $\mathcal{H}$. On peut représenter cet espace au moyen de l'opérateur linéaire K_T^* de l'ensemble des fonctions en escalier $\mathcal{E}$ dans $\mathrm{L}^2(0,T)$ défini pour φ dans $\mathcal{E}$ par

$$\left(K_T^*\varphi\right)(s)=\varphi(s)K(T,s)+\int_s^T(\varphi(t)-\varphi(s))K(dt,s).$$

Alors $\mathcal{H}$ est l'adhérence de $\mathcal{E}$ pour la norme $\|\varphi\|_{\mathcal{H}} = \|K_T^*\varphi\|_{\mathrm{L}^2(0,T)}$. Pour des fonctions φ de $\mathcal{E}$ et h de $\mathrm{L}^2(0,T)$ nous avons

$$\int_0^T \left(K_T^*\varphi\right)(t)h(t)dt = \int_0^T \varphi(t)(Kh)(dt).$$

Cette dualité permet d'étendre l'intégrale par rapport à $Kh(dt)$, définie pour les fonctions en escalier, à des intégrandes de $\mathcal{H}$. Mais aussi enfin de définir une intégrale stochastique de type Skohorod en définissant, pour des intégrandes φ dans $\mathcal{H}$,

$$\delta^X(\varphi) = \int_0^T \left(K_T^*\varphi\right)(t)\delta\beta(t) = \int_0^T \left(K_T^*\varphi\right)(t)d\beta(t).$$

Pour des intégrandes adaptées l'intégrale de Skohorod s'écrit donc comme une intégrale stochastique usuelle pour le mouvement Brownien.

Nos intégrandes seront déterministes et nous envisagerons des intégrandes à valeurs dans un espace de Hilbert comme cela est fait dans [137]. Elle revient à définir K_T^* comme un opérateur sur des fonctions à valeurs un espace de Hilbert avec l'expression ci-dessus. Dans ce cas, l'intégrale est une intégrale au sens de Bochner.

La relation de dualité est encore vraie pour des fonctions en escalier à valeurs dans un espace de Hilbert. L'opérateur K_T^* commute avec le produit scalaire sur l'espace de Hilbert. Ainsi le produit scalaire de l'intégrale stochastique est l'intégrale stochastique dans $\mathbb{R}$ du produit scalaire. Lorsque l'intégrande est une famille d'opérateurs $(\Lambda(t))_{t\in[0,T]}$ d'un espace de Hilbert E dans un espace de Hilbert E' satisfaisant

$$\sum_{j\in\mathbb{N}} \int_0^T \|\left(K_T^*\Lambda(\cdot)\Phi e_j\right)(t)\|_{E'}^2 dt < \infty,$$

l'intégrale est définie pour t positif par

$$\int_0^t \Lambda(s)dW^H(s) = \sum_{j\in\mathbb{N}} \int_0^t \Lambda(s)\Phi e_j d\beta_j^H(s) = \sum_{j\in\mathbb{N}} \int_0^t \left(K_t^*\Lambda(\cdot)\Phi e_j\right)(s)d\beta_j(s).$$

Nous commençons alors par montrer le résultat suivant sur la convolution stochastique.

Lemme 2.5.1 *La convolution stochastique $Z: \ t \mapsto \int_0^t U(t-s)dW^H(s)$ est bien définie. De plus, sous l'hypothèse (A), elle admet une modification à trajectoires $\alpha-$Hölderiennes pour $\alpha < \frac{\gamma}{2} \wedge H$ si $\gamma > 0$ et seulement continues si $\gamma = 0$ (cela n'est possible que si $H > \frac{1}{2}$).*

Elle définit une variable aléatoire à valeurs dans $\mathrm{C}\left([0,\infty);\mathrm{H}^1\right)$. De plus, les mesures images directes $\mu^{Z,T}$ de μ^Z par la restriction sur $\mathrm{C}\left([0,T];\mathrm{H}^1\right)$ pour T positif sont des mesures Gaussiennes centrées.

Nous considérons par la suite une telle modification. Nous notons $v^{u_0}(z)$ la solution mild de

$$\left\{ \begin{array}{l} i\frac{\mathrm{d}v}{\mathrm{d}t} = \Delta v + \lambda |v - iz|^{2\sigma}(v - iz) \\ u(0) = u_0 \in \mathrm{H}^1 \end{array} \right. ,$$

où z appartient à $\mathrm{C}\left([0,\infty);\mathrm{H}^1\right)$. Le problème est localement bien posé grâce à un argument de point fixe dans $\mathrm{C}\left([0,T];\mathrm{H}^1\right)$ pour T suffisamment petit. Celui-ci utilise le fait que la non linéarité est Lipschitzienne sur les bornés de H^1.

Nous définissons alors les solutions comme des solutions dans l'espace de trajectoires explosives $\mathcal{E}(\mathrm{H}^1)$. Rappelons que nous commençons par ajouter un point Δ à l'espace H^1 et que l'on munit l'ensemble de la topologie telle que les ouverts sont ceux de H^1 et les complémentaires dans $\mathrm{H}^1 \cup \{\Delta\}$ des fermés bornés. L'ensemble $\mathrm{C}([0,\infty);\mathrm{H}^1 \cup \{\Delta\})$ est alors bien défini comme l'intersection des espaces $\mathrm{C}([0,T];\mathrm{H}^1 \cup \{\Delta\})$ pour T positif. Le temps d'explosion de f dans $\mathrm{C}([0,\infty);\mathrm{H}^1 \cup \{\Delta\})$ est alors $\mathcal{T}(f) = \inf\{t \in [0,\infty) : f(t) = \Delta\}$, avec la convention que $\inf \emptyset = \infty$. Nous pouvons alors définir

$$\mathcal{E}(\mathrm{H}^1) = \left\{ f \in \mathrm{C}([0,\infty);\mathrm{H}^1 \cup \{\Delta\}) : f(t_0) = \Delta \Rightarrow \forall t \geq t_0,\ f(t) = \Delta \right\},$$

muni de la topologie définie par la base de voisinages

$$V_{T,r}(\varphi_1) = \left\{ \varphi \in \mathcal{E}(\mathrm{H}^1) : \mathcal{T}(\varphi) > T,\ \|\varphi_1 - \varphi\|_{\mathrm{C}([0,T];\mathrm{H}^1)} \leq r \right\}$$

de φ_1 dans $\mathcal{E}(\mathrm{H}^1)$ pour $T < \mathcal{T}(\varphi_1)$ et r positif. C'est un espace séparé.

Si nous posons $\mathcal{G}^{u_0}$ l'application

$$\mathcal{G}^{u_0}: \ z \mapsto v^{u_0}(z) - iz,$$

nous avons $u^{\epsilon,u_0} = \mathcal{G}^{u_0}(\sqrt{\epsilon} Z)$ où Z est la convolution stochastique.

Nous notons $(\mathcal{F}_t)_{t\geq 0}$ la filtration engendrée par le processus de Wiener fractionnaire.

Nous avons alors les deux résultats qui suivent.

Lemme 2.5.2 *L'application*

$$\begin{array}{rcl} \mathrm{C}\left([0,\infty);\mathrm{H}^1\right) & \to & \mathcal{E}\left(\mathrm{H}^1\right) \\ z & \mapsto & \mathcal{G}^{u_0}(z) \end{array}$$

est continue.

Théorème 2.5.1 *Sous l'hypothèse* (A) *et pour des données initiales* u_0 $\mathcal{F}_0$ *mesurables à valeurs dans* H^1*, il existe une unique solution mild au problème de Cauchy continue à valeurs dans* H^1*. Elle est définie sur un intervalle de temps aléatoire* $[0,\tau^*(u_0,\omega))$ *où* $\tau^*(u_0,\omega)$ *est un temps d'arrêt tel que*

$$\tau^*(u_0,\omega)=\infty \text{ or } \lim_{t\to\tau^*(u_0,\omega)}\|u(t)\|_{\mathrm{H}^1}=\infty.$$

En outre, τ^* *est presque sûrement semi continue par rapport à* u_0*. La solution* u *définit une variable aléatoire à valeurs dans* $\mathcal{E}\left(\mathrm{H}^1\right)$*.*

Lemme 2.5.3 *L'opérateur de covariance de* Z *sur* $\mathrm{L}^2\left(0,T;\mathrm{L}^2\right)$ *est donné pour* h *dans* $\mathrm{L}^2\left(0,T;\mathrm{L}^2\right)$ *par*

$$\begin{array}{rl} \mathcal{Q}h(t)=\sum_{j\in\mathbb{N}}\int_0^T\int_0^{t\wedge u} & \left(K_T^*\mathbb{1}_{[0,t]}(\cdot)U(t-\cdot)\Phi e_j\right)(s) \\ & \left(\left(K_T^*\mathbb{1}_{[0,u]}(\cdot)U(u-\cdot)\Phi e_j\right)(s),h(u)\right)_{\mathrm{L}^2}dsdu, \end{array}$$

lorsque $H>\frac{1}{2}$ *nous pouvons écrire* $\mathcal{Q}h(t)$ *comme*

$$c_H^2\left(H-\frac{1}{2}\right)^2B\left(2-2H,H-\frac{1}{2}\right)\int_0^T\int_0^t\int_0^s|u-v|^{2H-2}U(t-v)\Phi\Phi^*U(u-s)h(s)dudvds,$$

où B *est la fonction Beta.*

L'espace de Hilbert noyau auto reproduisant de $\mu^{Z,T}$ *est* $\mathrm{Im}\ \mathcal{Q}^{\frac{1}{2}}$ *avec la norme de la structure image. Il vaut également* $\mathrm{Im}\mathcal{L}$ *où* $\mathcal{L}$ *est défini pour* h *dans* $\mathrm{L}^2\left(0,T;\mathrm{L}^2\right)$ *par*

$$\mathcal{L}h(t)=\sum_{j\in\mathbb{N}}\int_0^t\left(K_T^*\mathbb{1}_{[0,t]}(\cdot)U(t-\cdot)\Phi e_j\right)(s)(h(s),e_j)_{\mathrm{L}^2}ds.$$

De plus les mesures images directes pour ϵ *positif de* $x\mapsto\sqrt{\epsilon}x$ *sur* $\mathrm{C}\left([0,\infty);\mathrm{H}^1\right)$ *satisfont un PGD de vitesse* ϵ *et de bonne fonction de taux*

$$I^Z(f)=\frac{1}{2}\inf_{h\in\mathrm{L}^2(0,\infty;\mathrm{L}^2):\mathcal{L}(h)=f}\left\{\|h\|^2_{\mathrm{L}^2(0,\infty;\mathrm{L}^2)}\right\}.$$

Nous en déduisons les résultats suivants.

Soit $\mu^{u^{\epsilon,u_0}}$ les lois sur $\mathcal{E}(\mathrm{H}^1)$ des solutions mild u^{ϵ,u_0} de

$$\left\{\begin{array}{l} i\mathrm{d}u-(\Delta u+\lambda|u|^{2\sigma}u)\mathrm{d}t=\sqrt{\epsilon}\mathrm{d}W^H, \\ u(0)=u_0 \ \in \mathrm{H}^1. \end{array}\right. \tag{2.5.2}$$

Nous avons le PGD suivant.

Théorème 2.5.2 *Les lois $\mu^{u^{\epsilon,u_0}}$ sur $\mathcal{E}(\mathrm{H}^1)$ satisfont un PGD de vitesse ϵ et de bonne fonction de taux*

$$I^{u_0}(w)=\frac{1}{2}\inf_{h\in \mathrm{L}^2(0,\infty;\mathrm{L}^2):\mathbf{S}(u_0,h)=w}\left\{\|h\|^2_{\mathrm{L}^2(0,\infty;\mathrm{L}^2)}\right\},$$

où $\mathbf{S}(u_0,h)$, le squelette, est la solution mild dans $\mathcal{E}(\mathrm{H}^1)$ du problème de contrôle

$$\left\{\begin{array}{l} i\frac{\partial u}{\partial t}-(\Delta u+\lambda|u|^{2\sigma}u)=\Phi\dot{K}h, \\ u(0)=u_0 \ \in \mathrm{H}^1 \\ h\in \mathrm{L}^2\left(0,\infty;\mathrm{L}^2\right); \end{array}\right.$$

Seule l'intégrale, ou celle de la forme mild, du membre de droite est définie à partir de la relation de dualité.

Nous obtenons alors avec les mêmes arguments que dans [81] le théorème de support.

Théorème 2.5.3 *Le support de la loi $\mu^{u^{1,u_0}}$ sur $\mathcal{E}(\mathrm{H}^1)$ est donné par*

$$supp\ \mu^{u^{1,u_0}}=\overline{\mathrm{Im}\mathcal{L}}^{\mathcal{E}(\mathrm{H}^1)}.$$

Chapitre 3

Conclusion et perspectives

Nous avons étudié plusieurs aspects des grandes déviations lorsque le bruit tend vers 0 pour des équations de Schrödinger non linéaires stochastiques. Nous nous sommes intéressés à des principes de grandes déviations trajectoriels. Nous avons traité des non linéarités sous critiques et surcritiques et déduit des résultats sur l'explosion en temps fini. Nous nous sommes aussi intéressés à des applications en physique et avons appliqué nos résultats à l'étude de l'erreur de transmission par solitons dans les fibres optiques. Nous avons obtenu de manière rigoureuse plusieurs résultats de physique dont les preuves semblent difficiles à justifier mathématiquement. Certains résultats sont nouveaux. Nous nous sommes intéressés à l'étude des temps et des points de sortie d'un domaine d'équilibre pour des équations faiblement amorties. Enfin nous avons commencé à étudier le cas de bruits Gaussiens plus généraux.

Plusieurs approfondissements seraient possibles.

Tout d'abord, nous avons vu que l'étude de l'asymptotique des queues pour des bruits tendant vers 0 est reliée à un problème de contrôle optimal. Les queues de la position sont plus épaisses que celles de la masse pour de longues fibres. Le risque de voir la position excéder un seuil (relativement petit par rapport à la longueur T de la fibre) est supérieur à celui que la masse excède un seuil. Néanmoins il est possible de concevoir des fibres avec éléments de contrôle tels que les queues, en particulier de la position, soient moins épaisses. Dans [128] il est suggéré que du point de vue de l'ingénierie il faille optimiser sur de tels champs externes tout en satisfaisant une contrainte de coût. Le nouveau problème de contrôle optimal nécessite une double optimisation. Nous pourrions alors diminuer exponentiellement les fluctuations non désirées. Dans [63], différents éléments de contrôle réduisant la fluctuation du centre sont suggérés. Par contre il n'est pas proposé d'optimiser sur ces éléments afin d'en définir avec des propriétés optimales.

Cette même idée peut permettre de réduire ou d'augmenter le temps moyen de sortie d'un domaine d'attraction ou d'agir sur le point de sortie. Lorsque plusieurs points d'équilibre existent cette technique peut permettre de rendre plus ou moins fréquentes certaines transitions mais aussi de définir a priori la forme des transitions.

Il est possible également de trancher la question que se posent dans beaucoup de papiers les physiciens sur le caractère Gaussien ou non de la loi du centre. Nous avons obtenu que sur une échelle logarithmique les queues sont les mêmes que celle d'une loi exponentielle. La différence se situerait au niveau des facteurs pré-exponentiels. Ceux-ci peuvent être obtenus par des grandes déviations précises. On pourrait donc chercher à montrer des grandes déviations précises pour les équations de equations de Schrödinger non linéaires stochastiques et s'inspirer notamment de [6, 7, 14, 121, 122, 123, 124].

Il serait aussi très intéressant de regarder de plus près le résultat sur la sortie d'un domaine dans H^1. En effet, nous avons vu que pour des bruits additifs celui-ci semble intimement lié aux ondes solitaires. Il conviendrait également de chercher à résoudre les problèmes de contrôle qui subsistent.

Mais aussi, pour certains systèmes purement Hamiltoniens, ce qui est le cas pour les équations sans amortissement, l'asymptotique de l'espérance du premier temps de sortie de domaines délimités par des ensembles de niveau de l'Hamiltonien ou de toute autre quantité invariante est souvent obtenu après changement de temps, application de la formule d'Itô aux quantités invariantes évaluées en la solution aléatoire et la technique de "stochastic averaging" (voir par exemple [69, 70, 71, 72, 73, 74, 133, 134, 135]). Le changement de temps est tel que le mouvement lent entre les différents niveaux des quantités invariantes du flot déterministe soit de vitesse 1. Les fluctuations rapides s'effectuant le long des ensembles de niveau. En général l'échelle de temps n'est plus exponentielle. Il serait intéressant de regarder ce type de problème pour des équations de Schrödinger non-linéaires stochastiques.

Par ailleurs plusieurs résultats de stabilité orbitale (résultats de stabilité tenant compte des groupes de symétrie pour l'équation), *c.f.* par exemple [26, 91, 127, 140], ou de stabilité asymptotique, *c.f.* par exemple [19, 20, 31, 32, 75, 130], existent pour les équations de Schrödinger non linéaires. Il serait très intéressant d'étudier l'influence d'un petit bruit sur un résultat de stabilité asymptotique, mais ces résultats sont partiels et par exemple valables pour certaines données initiales seulement, ils reposent sur des hypothèses sur l'opérateur linéarisé au voisinage de l'onde solitaire...

On pourrait chercher à prouver des grandes déviations pour les mesures invariantes lorsque le bruit tend vers 0 comme cela est fait dans [28, 73, 132]. Des résultats sur les mesures invariantes pour une équation de Schrödinger non linéaire stochastique avec non-linéarité cubique et focalisante sur un domaine borné et avec un bruit additif ont été prouvés dans [45].

Il serait aussi possible d'étudier des bruits multiplicatifs relativement généraux en adoptant une approche basée sur les trajectoires rugueuses. Il conviendrait alors de montrer un résultat de continuité par rapport aux trajectoires rugueuses du processus dirigeant l'équation analogue à celui de [107] pour les EDS. Nous pourrions en déduire des principes de grandes déviations et des théorèmes de support en procédant comme dans [103].

Enfin, nous pourrions aussi nous intéresser à des grandes déviations pour la famille de mesures d'occupation

$$\left(\frac{1}{T}\int_0^T \mathbb{1}_B(u^{u_0}(s))ds\right)_{T>0}$$

où B est un borélien de L^2 ou H^1 et plus généralement pour la famille de mesures empiriques

$$\left(\frac{1}{T}\int_0^T \delta_{u^{u_0}(s)}ds\right)_{T>0}$$

où u^{u_0} est la solution de l'équation stochastique issue de u_0. Les principes de grandes déviations sont alors des principes de grandes déviations de niveau 2 car les lois des mesures aléatoires ci-dessus sont des mesures sur des espaces de mesures. Ils quantifient un résultat de convergence faible vers la mesure de Dirac en la mesure invariante. Pour cela il est possible de se référer à [48] pour le cas des chaînes de Markov, à [53] pour des résultats plus généraux et à [30, 96, 115, 120] pour des diffusions . On peut aussi se référer aux articles de Donsker et Varadhan [54, 55, 56, 57] et à l'application [58] au problème du Polaron.

Appendix A

Large deviations and support for nonlinear Schrödinger equations with additive noise and applications

Abstract: Sample path large deviations for the laws of the solutions of stochastic nonlinear Schrödinger equations when the noise converges to zero are presented. The noise is a complex additive Gaussian noise. It is white in time and colored in space. The solutions may be global or blow-up in finite time, the two cases are distinguished. The results are stated in trajectory spaces endowed with projective limit topologies. In this setting, the support of the law of the solution is also characterized. As a consequence, results on the law of the blow-up time and asymptotics when the noise converges to zero are obtained. An application to the transmission of solitary waves in fiber optics is also given.

A.1 Introduction

In this article, the stochastic nonlinear Schrödinger (NLS) equation with a power law nonlinearity and an additive noise is studied. The deterministic equation occurs as a basic model in many areas of physics: hydrodynamics, plasma physics, nonlinear optics, molecular biology. It describes the propagation of waves in media with both nonlinear and dispersive responses. It is an idealized model and does not take into account many aspects such as inhomogeneities, high order terms, thermal fluctuations, external forces which may be modeled as a random excitation (see [50, 59, 62, 63, **?**, 114]). Propagation in random media may also be considered. The resulting re-scaled equation is a random perturbation of the dynamical system of the following form:

$$i\frac{\partial}{\partial t}\psi - (\Delta\psi + \lambda|\psi|^{2\sigma}\psi) = \xi, \quad x \in \mathbb{R}^d, \quad t \geq 0, \quad \lambda = \pm 1, \tag{A.1.1}$$

where ξ is a complex valued space-time white noise with correlation function, following the notation used in [62],

$$\mathbb{E}\left[\xi(t_1, x_1)\bar{\xi}(t_2, x_2)\right] = D\delta_{t_1 - t_2} \otimes \delta_{x_1 - x_2}$$

D is the noise amplitude and δ denotes the Dirac mass. When $\lambda = 1$ the nonlinearity is called focusing, otherwise it is defocusing.

With the notations of the next section, the unbounded operator $-i\Delta$ on L^2 with domain H^2 is skew-adjoint. Stone's theorem gives thus that it generates a unitary group $(S(t) = e^{-it\Delta})_{t\in\mathbb{R}}$. The Fourier transform gives that this group is also unitary on every Sobolev space based on L^2. Consequently, there is no smoothing effect in the Sobolev spaces. We are thus unable to treat the space-time white noise and will consider a complex valued centered Gaussian noise, white in time and colored in space.

In the present article, the formalism of stochastic evolution equations in Banach spaces as presented in [34] is adopted. This point of view is preferred to the field and martingale measure stochastic integral approach, see [139], in order to use a particular property of the group, namely hypercontractivity. The Strichartz inequalities, presented in the next section, show that some integrability property is gained through time integration and "convolution" with the group. In this setting, the Gaussian noise is defined as the time derivative in the sense of distributions of a Q-Wiener process $(W(t))_{t\in[0,\infty)}$ on H^1. Here Q is the covariance operator of the law of the H^1-random variable $W(1)$, which is a centered Gaussian measure. With the Itô notations, the stochastic evolution equation is written

$$i\mathrm{d}u - (\Delta u + \lambda|u|^{2\sigma}u)\mathrm{d}t = \mathrm{d}W. \tag{A.1.2}$$

The initial datum u_0 is a function of H^1. We will consider solutions of NLS that are weak solutions in the sense used in the analysis of partial differential equations or equivalently mild solutions which satisfy

$$u(t) = S(t)u_0 - i\lambda\int_0^t S(t-s)(|u(s)|^{2\sigma}u(s))ds - i\int_0^t S(t-s)dW(s). \tag{A.1.3}$$

The well posedness of the Cauchy problem associated to (A.1.1) in the deterministic case depends on the size of σ. If $\sigma < \frac{2}{d}$, the nonlinearity is subcritical and the Cauchy problem is globally well posed in L^2 or H^1. If $\sigma = \frac{2}{d}$, critical nonlinearity, or $\frac{2}{d} < \sigma < \frac{2}{d-2}$ when $d \geq 3$ or simply $\sigma > \frac{2}{d}$ otherwise, supercritical nonlinearity, the Cauchy problem is locally well posed in H^1; see [99]. In this latter case, if the nonlinearity is defocusing, the solution is global. In the focusing case some initial data yield global solutions while it is known that other initial data yield solutions which blow up in finite time; see [25, 136].
In [37], the H^1 results have been generalized to the stochastic case and existence and uniqueness results are obtained for the stochastic equation under the same conditions on σ. Continuity with respect to the initial data and the perturbation is proved. It is shown that the proof of global existence for a defocusing nonlinearity or for a focusing nonlinearity with a subcritical exponent, could be adapted in the stochastic case even if the mass

$$\mathbf{N}\left(u(t)\right) = \|u(t)\|_{\mathrm{L}^2}^2$$

and Hamiltonian

$$\mathbf{H}\left(u(t)\right) = \frac{1}{2}\int_{\mathbb{R}^d} \left|\nabla u(t,x)\right|^2 dx - \frac{\lambda}{2\sigma+2}\int_{\mathbb{R}^d} \left|u(t,x)\right|^{2\sigma+2} dx$$

are no longer conserved. For a focusing nonlinearity and critical or supercritical exponents, the solution may blow-up in finite time. The blow-up time is denoted by $\tau(\omega)$. It satisfies either $\lim_{t\to\tau(\omega)} \|u(t)\|_{\mathrm{H}^1} = \infty$ or $\tau(\omega) = \infty$, even if the solution is obtained by a fixed point argument in a ball of a space of more regular functions than $\mathrm{C}([0,T];\mathrm{H}^1)$.

In this article, we are interested in the law of the paths of the random solution. When the noise converges to zero, continuity with respect to the perturbation gives that the law converges to the Dirac mass on the deterministic solution. In the following, a large deviation result is shown. It gives the rate of convergence to zero, on a logarithmic scale, of the probability that paths are in sets that do not contain the deterministic solution. A general result is stated for the case where blow-up in finite time is possible and a second one for the particular case where the solutions are global. Also, the stronger the topology, the sharper are the estimates. We will therefore take advantage of the variety of spaces that can be considered for the fixed point argument, due to the integrability property, and present the large deviation principles in trajectory spaces endowed with projective limit topologies. A characterization of the support of the law of the solution in these trajectory spaces is proved. The two results can be transferred to weaker topologies or more generally by any continuous mapping. The first application is a proof that, for certain noises, with positive probability some solutions blow up after any time T. Some estimates on the law of the blow-up time when the noise converges to zero are also obtained. This study is yet another contribution to the study of the influence of a noise on the blow-up of the solutions of the focusing supercritical NLS; see in the case of an additive noise [38, 40]. A second application is given. It consists in obtaining similar results as in [62] with an approach based on large deviations. The aim is to compute estimates of error probability in signal transmission in optical fibers when the medium is random and nonlinear, for small noises. Uniform large deviations for small noise asymptotics when the noise enters linearly as a random potential the NLS equation are studied in [82]. In that case we had to use a more elaborate proof based on the Freidlin and Wentzell inequality and the continuity of the skeleton with respect to the control on the sets levels of the rate function of the initial Wiener process less or equal to a positive constant since in that case the Itô map fails to be continuous at the level of paths for the topologies we consider.

Section A.2 is devoted to notations and properties of the group, of the noise and of the stochastic convolution. An extension of the result of continuity with respect to the stochastic convolution presented in [37] is also given. In Section A.3, the large deviation principles (LDP) is presented. Section A.4 is devoted to the support result and the two last sections to the applications.

A.2 Notations and preliminary results

Throughout the paper the following notations will be used.

The set of positive integers and positive real numbers are denoted respectively by $\mathbb{N}^*$ and $\mathbb{R}_+^*$, while the set of real numbers different from 0 is denoted by $\mathbb{R}^*$.

For p in $\mathbb{N}^*$, L^p is the classical Lebesgue space of complex valued functions and $\mathrm{W}^{1,p}$ is the associated Sobolev space of L^p functions with first order derivatives, in the sense of distributions, in L^p. When $p = 2$, H^s denotes the fractional Sobolev space of tempered distributions $v \in \mathcal{S}'$ such that the Fourier transform $\hat{v}$ satisfies $(1 + |\xi|^2)^{s/2}\hat{v} \in \mathrm{L}^2$. The space L^2 is endowed with the inner product defined by $(u, v)_{\mathrm{L}^2} = \mathfrak{Re} \int_{\mathbb{R}^d} u(x)\overline{v}(x)dx$. Also, when it is clear that μ is a Borel measure on a specified Banach space, we simply write $\mathrm{L}^2(\mu)$ and do not specify the Banach space and Borel $\sigma-$field.

If I is an interval of $\mathbb{R}$, $(E, \|\cdot\|_E)$ a Banach space and r belongs to $[1,\infty]$, then $\mathrm{L}^r(I;E)$ is the space of strongly Lebesgue measurable functions f from I into E such that $t \to \|f(t)\|_E$ is in $\mathrm{L}^r(I)$. Let $\mathrm{L}^r_{loc}(0,\infty;E)$ be the respective spaces of locally integrable functions on $(0,\infty)$. They are endowed with topologies of Fréchet space. The spaces $\mathrm{L}^r(\Omega;E)$ are defined similarly.

We recall that a pair (r,p) of positive numbers is called an admissible pair if p satisfies $2 \le p < \frac{2d}{d-2}$ when $d > 2$ ($2 \le p < \infty$ when $d = 2$ and $2 \le p \le \infty$ when $d = 1$) and r is such that $\frac{2}{r} = d\left(\frac{1}{2} - \frac{1}{p}\right)$. For example $(\infty, 2)$ is an admissible pair.
When E is a Banach space, we will denote by E^* its topological dual space. For $x^* \in E^*$ and $x \in E$, the duality will be denoted $< x^*, x >_{E^*,E}$.
We recall that Φ is a Hilbert Schmidt operator from a Hilbert space H into a Hilbert space $\tilde{H}$ if it is a linear continuous operator such that, given a complete orthonormal system $(e_j^H)_{j\in\mathbb{N}}$ of H, $\sum_{j\in\mathbb{N}} \|\Phi e_j^H\|^2_{\tilde{H}} < \infty$. We will denote by $\mathcal{L}_2(H,\tilde{H})$ the space of Hilbert Schmidt operators from H into $\tilde{H}$ endowed with the norm

$$\|\Phi\|_{\mathcal{L}_2(H,\tilde{H})} = \mathrm{tr}\,(\Phi\Phi^*) = \sum_{j\in\mathbb{N}} \|\Phi e_j^H\|^2_{\tilde{H}},$$

where Φ^* denotes the adjoint of Φ and tr the trace. We denote by $\mathcal{L}_2^{s,r}$ the corresponding space for $H = \mathrm{H}^s$ and $\tilde{H} = \mathrm{H}^r$. In the introduction Φ has been taken in $\mathcal{L}_2^{0,1}$.

When A and B are two Banach spaces, $A \cap B$, where the norm of an element is defined as the maximum of the norm in A and in B, is a Banach space. The following Banach spaces defined for the admissible pair $(r(p),p)$ and positive T by

$$X^{(T,p)} = \mathrm{C}\left([0,T];\mathrm{H}^1\right) \cap \mathrm{L}^{r(p)}\left(0,T;\mathrm{W}^{1,p}\right)$$

will be of particular interest.

The probability space will be denoted by $(\Omega, \mathcal{F}, \mathbb{P})$. Also, $x \wedge y$ stands for the minimum of the two real numbers x and y and $x \vee y$ for the maximum. We recall that a rate function I is a lower semicontinuous function and that a good rate function I is a rate function such that for every $c > 0$, $\{x : I(x) \le c\}$ is a compact set. Finally, we will denote by *supp* μ the support of a probability measure μ on a topological vector space. It is the complement of the largest open set of null measure.

A.2.1 Properties of the group

When the group acts on the Schwartz space $\mathcal{S}$, the Fourier transform gives the following analytic expression

$$\forall u_0 \in \mathcal{S},\ \forall t \neq 0,\ S(t)u_0 = \frac{1}{(4i\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-i\frac{|x-y|^2}{4t}} u_0(y)dy.$$

The Fourier transform also gives that the adjoint of $S(t)$ in L^2 and in every Sobolev space on L^2 is $S(-t)$, the same bounded operator with time reversal.

The Strichartz inequalities, see [99], are the following

(i) There exists C positive such that for u_0 in H^1, T positive and $(r(p), p)$ admissible pair,

$$\|U(\cdot)u_0\|_{X^{(T,p)}} \leq C \|u_0\|_{\mathrm{L}^2},$$

(ii) For every T positive, $(r(p), p)$ and $(r(q), q)$ admissible pairs, s and ρ such that $\frac{1}{s} + \frac{1}{r(q)} = 1$ and $\frac{1}{\rho} + \frac{1}{q} = 1$, there exists C positive such that for f in $\mathrm{L}^s\left(0, T; \mathrm{W}^{1,\rho}\right)$,

$$\left\|\int_0^{\cdot} U(\cdot - s)f(s)ds\right\|_{X^{(T,p)}} \leq C\|f\|_{\mathrm{L}^s(0,T;\mathrm{W}^{1,\rho})}.$$

Remark A.2.1 *The first estimate gives the integrability property of the group, the second gives the integrability of the convolution that allows to treat the nonlinearity.*

A.2.2 Topology and trajectory spaces

Let us introduce a topological space that allows us to treat the subcritical case or the defocusing case. When $d > 2$, we set

$$\mathcal{X}_\infty = \bigcap_{T \in \mathbb{R}_+^*,\ 2 \leq p < \frac{2d}{d-2}} X^{(T,p)},$$

it is endowed with the projective limit topology; see [15] and [48]. When $d = 2$ and $d = 1$ we write $p \in [2, \infty)$.

The set of indices $\left(\mathbb{R}_+^* \times \left[2, \frac{2}{d-2}\right), \prec\right)$ when $d > 2$ or $\left(\mathbb{R}_+^* \times [2, \infty), \prec\right)$ when $d = 2$ or $d = 1$, where $(T, p) \prec (S, q)$ if $T \leq S$ and $p \leq q$, is a partially ordered right-filtering set.

If $(T,p) \prec (S,q)$ and $u \in X^{(S,q)}$, Hölder's inequality gives that for α such that $\frac{1}{p} = \frac{\alpha}{q} + \frac{1-\alpha}{2}$,

$$\|u(t)\|_{\mathrm{L}^p} \leq \|u(t)\|_{\mathrm{L}^2}^{1-\alpha} \|u(t)\|_{\mathrm{L}^q}^{\alpha}.$$

Consequently,

$$\|u(t)\|_{\mathrm{W}^{1,p}} \leq (d+1)\|u(t)\|_{\mathrm{H}^1}^{1-\alpha} \|u(t)\|_{\mathrm{W}^{1,q}}^{\alpha}.$$

By time integration, along with Hölder's inequality, the fact that $r(q) = \alpha r(p)$ and that $T \leq S$, we obtain that u is a function of $X^{(T,p)}$ and

$$\|u\|_{X^{(T,p)}} \leq (d+1)\|u\|_{X^{(S,q)}}. \tag{A.2.1}$$

If we denote by $p_{(S,q)}^{(T,p)}$ the dense and continuous embeddings from $X^{(S,q)}$ into $X^{(T,p)}$, they satisfy the consistency conditions

$$\forall\ (T,p) \prec (S,q) \prec (R,r),\ p_{(R,r)}^{(T,p)} = p_{(R,r)}^{(S,q)} \circ p_{(S,q)}^{(T,p)}.$$

Consequently, the projective limit topology is well defined by the following neighborhood basis, given for φ_1 in $\mathcal{X}_\infty$ by

$$U(\varphi_1;(T,p);\epsilon) = \left\{ \varphi \in \bigcap_{(T',p') \in J} X^{(T',p')} : \|\varphi - \varphi_1\|_{X^{(T,p)}} < \epsilon \right\}.$$

It is the weakest topology on the intersection such that for every $(T,p) \in J$, the injection $p_{(T,p)} : \mathcal{X}_\infty \to X^{(T,p)}$ is continuous. It is a standard fact, see [15], that $\mathcal{X}_\infty$ is a Hausdorff topological space.

Following from (A.2.1), a countable neighborhood basis of φ_1 is given by $\left(U\left(\varphi_1;(n,p(l));\frac{1}{k}\right)\right)_{(n,k,l)\in(N^*)^3}$, where $p(l) = 2 + \frac{4}{d-2} - \frac{1}{l}$ and $l > \frac{d-2}{4}$ if $d > 2$. If $d = 2$ and $d = 1$, we take $p(l) = l$.

Also it is convenient, for measurability issues, to note that $\mathcal{X}_\infty$ can be turned into a complete separable metric space, i.e. a Polish space, setting

$$\forall (x,y) \in \mathcal{X}_\infty^2,\ d(x,y) = \sum_{n > \frac{d-2}{4}} \frac{1}{2^n} \left(\|x-y\|_{X^{(n,p(n))}} \wedge 1 \right).$$

It can be checked that it is also a locally convex Fréchet space.

The following spaces are introduced for the case where blow-up may occur. Adding a point Δ to the space H^1 and adapting slightly the proof of Alexandroff's compactification, it can be seen that the open sets of H^1 and the complement in $\mathrm{H}^1\cup\{\Delta\}$ of the closed bounded sets of H^1 define the open sets of a topology on $\mathrm{H}^1\cup\{\Delta\}$. This topology induces on H^1 the topology of H^1. Also, with such a topology $\mathrm{H}^1\cup\{\Delta\}$ is a Hausdorff topological space. Note that in [5], where diffusions are studied, the compactification of $\mathbb{R}^d$ is considered. Nonetheless, compactness is not an important feature and the above construction is enough for the following.

The space $\mathrm{C}([0,\infty);\mathrm{H}^1\cup\{\Delta\})$ is the space of continuous functions with value in $\mathrm{H}^1\cup\{\Delta\}$. Also, if f belongs to $\mathrm{C}([0,\infty);\mathrm{H}^1\cup\{\Delta\})$ we denote the blow-up time by

$$\mathcal{T}(f)=\inf\{t\in[0,\infty):f(t)=\Delta\}.$$

As in [5], a space of exploding paths, where Δ acts as a cemetery, is introduced. We set

$$\mathcal{E}(\mathrm{H}^1)=\left\{f\in\mathrm{C}([0,\infty);\mathrm{H}^1\cup\{\Delta\}):\ f(t_0)=\Delta\Rightarrow\forall t\geq t_0,\ f(t)=\Delta\right\}.$$

It is endowed with the topology defined by the following neighborhood basis given for φ_1 in $\mathcal{E}(\mathrm{H}^1)$ by

$$V_{T,\epsilon}(\varphi_1)=\left\{\varphi\in\mathcal{E}(\mathrm{H}^1):\mathcal{T}(\varphi)\geq T,\ \|\varphi_1-\varphi\|_{\mathrm{L}^\infty([0,T];\mathrm{H}^1)}\leq\epsilon\right\},$$

where $T<\mathcal{T}(\varphi_1)$ and $\epsilon>0$.

As a consequence of the topology of $\mathcal{E}(\mathrm{H}^1)$, the function $\mathcal{T}:\mathcal{E}(\mathrm{H}^1)\to[0,\infty]$ is sequentially lower semicontinuous, this is to say that if a sequence of functions $(f_n)_{n\in\mathbb{N}}$ converges to f then $\underline{\lim}_{n\to\infty}\mathcal{T}(f_n)\geq\mathcal{T}(f)$. Following from (A.2.1), the topology of $\mathcal{E}(\mathrm{H}^1)$ is also defined by the countable neighborhood basis given for $\varphi_1\in\mathcal{E}(\mathrm{H}^1)$ by $\left(V_{\mathcal{T}(\varphi_1)-\frac{1}{n},\frac{1}{k}}(\varphi_1)\right)_{(n,k)\in(\mathbb{N}^*)^2}$. Therefore $\mathcal{T}$ is a lower semicontinuous mapping.

Note that, as topological spaces, the two following spaces satisfy the identity

$$\left\{f\in\mathcal{E}(\mathrm{H}^1):\ \mathcal{T}(f)=\infty\right\}=\mathrm{C}([0,\infty);\mathrm{H}^1).$$

Finally, the analogue of the intersection in the subcritical case endowed with projective limit topology is defined, when $d>2$, by

$$\mathcal{E}_\infty=\left\{f\in\mathcal{E}(\mathrm{H}^1):\forall\ p\in\left[2,\frac{2d}{d-2}\right),\ \forall\ T\in[0,\mathcal{T}(f)),\ f\in\mathrm{L}^{r(p)}\left(0,T;\mathrm{W}^{1,p}\right)\right\}.$$

When $d=2$ and $d=1$ we write $p\in[2,\infty)$. It is endowed with the topology defined for φ_1 in $\mathcal{E}_\infty$ by the following neighborhood basis

$$W_{T,p,\epsilon}(\varphi_1)=\{\varphi\in\mathcal{E}_\infty:\mathcal{T}(\varphi)\geq T,\ \|\varphi_1-\varphi\|_{X^{(T,p)}}\leq\epsilon\}.$$

where $T < \mathcal{T}(\varphi_1)$, p is as above and $\epsilon > 0$. From the same arguments as for the space $\mathcal{X}_\infty$, $\mathcal{E}_\infty$ is a Hausdorff topological space. Also, as previously, (A.2.1) gives that the topology can be defined for φ_1 in $\mathcal{E}_\infty$ by the countable neighborhood basis $\left(W_{\mathcal{T}(\varphi_1)-\frac{1}{n},p(n),\frac{1}{k}}(\varphi_1)\right)_{(n,k)\in(\mathbb{N}^*)^2:\ n>\frac{d-2}{4}}$.

If we denote again by $\mathcal{T} : \mathcal{E}_\infty \to [0,\infty]$ the blow-up time, since $\mathcal{E}_\infty$ is continuously embedded into $\mathcal{E}(\mathrm{H}^1)$, $\mathcal{T}$ is lower semicontinuous. Thus, since $\{[0,t],\ t \in [0,\infty]\}$ is a $\pi-$system that generates the Borel $\sigma-$algebra of $[0,\infty]$, $\mathcal{T}$ is measurable. Note also that, as topological spaces, the following spaces are identical

$$\{f \in \mathcal{E}_\infty : \mathcal{T}(f) = \infty\} = \mathcal{X}_\infty.$$

A.2.3 Statistical properties of the noise

The Q-Wiener process W is such that its trajectories are in $\mathrm{C}([0,\infty);\mathrm{H}^1)$. We assume in the following that $Q = \Phi\Phi^*$ where Φ is a Hilbert-Schmidt operator from L^2 into H^1. The Wiener process can therefore be written as $W = \Phi W_c$ where W_c is a cylindrical Wiener process.

We recall that for any orthonormal basis $(e_j)_{j\in\mathbb{N}}$ of L^2, there exists a sequence of real independent Brownian motions $(\beta_j)_{j\in\mathbb{N}}$ such that $W_c = \sum_{j\in\mathbb{N}}\beta_j e_j$. The sum $W_c = \sum_{j\in\mathbb{N}}\beta_j e_j$ is well defined in every Hilbert space H such that L^2 is embedded into H with a Hilbert Schmidt embedding. We say that it is cylindrical because it is such that the decomposition of $W_c(1)$ on cylinder sets $(e_1, ..., e_N)$ are the finite dimensional centered Gaussian variables $(\beta_1(1), ..., \beta_N(1))$ with a covariance equal to the identity. The law of $W(1)$ is thus the direct image measure by the Hilbert-Schmidt mapping Φ of the natural extension of the corresponding sequence of centered Gaussian measures in finite dimensions, with a covariance equal to identity. In other words it is the *bona-fide* $\sigma-$additive direct image measure of a Gaussian cylindrical measure. Also, formally, for T positive the coefficients of the series expansion of the derivative of W_c on the tensor product of the complete orthonormal systems of L^2 and of $\mathrm{L}^2(0,T)$, given for example by the time derivative of the eigenvectors of the correlation operator of the law on $\mathrm{C}([0,T])$ of the Brownian motions, is a sequence of independent real-valued standard normal random variables. It is thus a Gaussian white noise.

In reference [62] the authors define the correlation function by the quantity

$$\mathbb{E}\left[\frac{\partial}{\partial t}W_c(t+s,x+z)\overline{\frac{\partial}{\partial t}W_c(t,x)}\right],$$

writing down formally the series expansion we obtain in the case of the white noise $2\delta_0(s) \otimes \delta_0(z)$. In the case of our space-colored noise, we obtain the multiplication of $\delta_0(s)$ by the L^2 function $\sum_{j\in\mathbb{N}} \Phi e_j(x+z)\overline{\Phi e_j(x)}$, where $(e_j)_{j\in\mathbb{N}}$ is a complete orthonormal system of L^2. Also as the operator Φ belongs to $\mathcal{L}_2^{0,1}$ and thus to $\mathcal{L}_2^{0,0}$ it may be defined through the kernel $\mathcal{K}(x,y) = \frac{1}{2}\sum_{j\in\mathbb{N}} \Phi\overline{e_j}(x)e_j(y)$ of $\mathrm{L}^2(\mathbb{R}^d \times \mathbb{R}^d)$, considering that $(e_j)_{j\in\mathbb{N}}$ consists of $(f_j)_{j\in\mathbb{N}}$ a complete orthonormal system of $\mathrm{L}^2(\mathbb{R}^d,\mathbb{R})$ and of $(if_j)_{j\in\mathbb{N}}$. This means that for any square integrable function u, $\Phi u(x) = \int_{\mathbb{R}^d} \mathcal{K}(x,y)u(y)dy$. In that case we could write the correlation function

$$\left(2\int_{\mathbb{R}^d} \mathcal{K}(x+z,u)\overline{\mathcal{K}}(x,u)du\right)\delta_0(s).$$

In the following we assume that the probability space is endowed with the filtration $\mathcal{F}_t = \mathcal{N} \cup \sigma\{W_s, 0 \le s \le t\}$ where $\mathcal{N}$ denotes the $\mathbb{P}-$null sets.

A.2.4 The random perturbation

We define the stochastic convolution by $Z(t) = \int_0^t S(t-s)dW(s)$ and the operator $\mathcal{L}$ on $\mathrm{L}^2(0,T;\mathrm{L}^2)$ by

$$\mathcal{L}h(t) = \int_0^t I \circ S(t-s)\Phi h(s)ds,\ \ h \in \mathrm{L}^2(0,T;\mathrm{L}^2),$$

where I is the injection of H^1 into L^2.

Proposition A.2.2 *The stochastic convolution defines a measurable mapping from $(\Omega,\mathcal{F})$ into $\left(\mathcal{X}_\infty,\mathcal{B}^X\right)$, where $\mathcal{B}^X$ stands for the Borel $\sigma-$field. Its law is denoted by μ^Z.*
The direct images $\mu^{Z;(T,p)} = p_{(T,p)*}\mu^Z$ on the real Banach spaces $X^{(T,p)}$ are centered Gaussian measures of reproducing kernel Hilbert space (RKHS) $H_{\mu^{Z;(T,p)}} = \mathrm{Im}\mathcal{L}$ with the norm of the image structure.

Proof. Setting $F(t) = \int_0^t S(-u)dW(u)$, for $t \in \mathbb{R}^+$, $Z(t) = S(t)F(t)$ follows. Indeed, if $(f_j)_{j\in\mathbb{N}}$ is a complete orthonormal system of H^1, a straightforward calculation gives that $(Z(t),f_j)_{\mathrm{H}^1} = (S(t)F(t),f_j)_{\mathrm{H}^1}$ for every j in $\mathbb{N}$. The continuity of the paths in H^1 follows from the construction of the stochastic integral with respect to the Wiener process since the deterministic operator integrand satisfies $\int_0^T \|S(-u)\Phi\|^2_{\mathcal{L}_2^{0,1}} < \infty$ and from the strong continuity of the group.

Step 1: We claim that the mapping Z is measurable from $(\Omega,\mathcal{F})$ into

$\left(X^{(T,p)}, \mathcal{B}^{(T,p)}\right)$, where $\mathcal{B}^{(T,p)}$ denotes the associated Borel $\sigma-$field.

Since $X^{(T,p)}$ is a Polish space, every open set is a countable union of open balls and consequently $\mathcal{B}^{(T,p)}$ is generated by open balls. Note that the event $\{\omega \in \Omega : \|Z(\omega) - x\|_{X^{(T,p)}} \leq r\}$ is equal to

$$\begin{aligned}&\left(\bigcap_{s\in\mathbb{Q}\cap[0,T]} \{\omega \in \Omega : \|Z(s)(\omega) - x\|_{\mathrm{H}^1} \leq r\}\right)\\ &\bigcap \left\{\omega \in \Omega : \|Z(\omega) - x\|_{\mathrm{L}^{r(p)}(0,T;\mathrm{W}^{1,p})} \leq r\right\}.\end{aligned}$$

Also, note that, since $(Z(t))_{t\in\mathbb{R}^+}$ is a collection of H^1 random variables, the first part is a countable intersection of elements of $\mathcal{F}$. Consequently, it suffices to show that : $\omega \mapsto (t \mapsto Z(t))$ defines a $\mathrm{L}^{r(p)}(0,T;\mathrm{W}^{1,p})$ random variable.

Consider $(\Phi_n)_{n\in\mathbb{N}}$ a sequence of operators of $\mathcal{L}_2^{0,2}$ converging to Φ for the topology of $\mathcal{L}_2^{0,1}$ and Z_n the associated stochastic convolutions. The Sobolev injections along with Hölder's inequality give that when $d > 2$ and $2 \leq p \leq \frac{2d}{d-2}$, H^1 is continuously embedded in L^p. It also gives that, when $d = 2$, H^1 is continuously embedded in every L^p for every $p \in [2,\infty)$ and for every $p \in [2,\infty]$ when $d = 1$. Consequently, for every n in $\mathbb{N}$, Z_n defines a $\mathrm{C}([0,T];\mathrm{H}^2)$ random variable and therefore a $\mathrm{L}^{r(p)}\left(0,T;\mathrm{W}^{1,p}\right)$ random variable for the corresponding values of p.

Revisiting the proof of Proposition 3.1 in reference [37] and letting $2\sigma+2$ be replaced by any of the previous values of p besides $p = \infty$ when $d = 1$, the necessary measurability issues to apply the Fubini's theorem are satisfied. Also, one gets the same estimates and that there exists a constant $C(d,p)$ such that for every n and m in $\mathbb{N}$,

$$\mathbb{E}\left[\|Z_{n+m}(\omega) - Z_n(\omega)\|^r_{\mathrm{L}^{r(p)}(0,T;\mathrm{W}^{1,p})}\right] \leq C(d,p)T^{\frac{r}{2}-1}\|\Phi_{n+m} - \Phi_n\|^r_{\mathcal{L}_2^{0,1}}.$$

The sequence $(Z_n)_{n\in\mathbb{N}}$ is thus a Cauchy sequence of the Banach space $\mathrm{L}^r\left(\Omega;\mathrm{L}^r\left(0,T;\mathrm{W}^{1,p}\right)\right)$ and converges to $\tilde{Z}$. The previous calculation also gives that

$$\mathbb{E}\left[\|Z_n(\omega) - Z(\omega)\|^r_{\mathrm{L}^{r(p)}(0,T;\mathrm{L}^p)}\right] \leq C(d,p)T^{\frac{r}{2}-1}\|\Phi_n - \Phi\|^r_{\mathcal{L}_2^{0,1}}.$$

Therefore $\tilde{Z} = Z$, Z belongs to $\mathrm{L}^{r(p)}\left(0,T;\mathrm{W}^{1,p}\right)$ and it defines a measurable mapping as expected.

Note that in $\mathcal{X}_\infty$, to simplify the notations, we did not write the cases $p = \infty$ when $d = 1$ or $p = \frac{2d}{d-2}$ when $d > 2$. We are indeed interested in results on the laws of the solutions of stochastic NLS and not really on the stochastic convolution. Also, the result of continuity in the next section shows that we necessarily lose on p in order to interpolate with $2 < p < p'$ and have a nonzero exponent on the L^2-norm. Therefore, even if it seems at first glance that we lose on the Sobolev's injections, it is not a restriction.

Step 2: We show that the mapping Z is measurable with values in $\mathcal{X}_\infty$ with the Borel $\sigma-$field $\mathcal{B}^{\mathcal{X}_\infty}$.

From step 1, given $x \in \mathcal{X}_\infty$, for every n in $\mathbb{N}^*$ such that $n > \frac{d-2}{4}$ the mapping $\omega \mapsto \|Z(\omega) - x\|_{X^{(n,p(n))}}$ from $(\Omega, \mathcal{F})$ into $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$, where $\mathcal{B}(\mathbb{R}^+)$ stands for the Borel $\sigma-$field of $\mathbb{R}^+$, is measurable. Thus

$$\omega \mapsto d(Z(\omega), x) = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{2^n} \left(\|Z(\omega) - x\|_{X^{(n,p(n))}} \wedge 1\right)$$

is measurable. Consequently, for every r in $\mathbb{R}^+$, $\{\omega \in \Omega \ : \ d(Z(\omega), x) < r\}$ belongs to $\mathcal{F}$.

Note that the law $\mu^{Z;\mathcal{X}_\infty}$ of Z on the metric space $\mathcal{X}_\infty$, which is a positive Borel measure, is therefore also regular and consequently it is a Radon measure.

Step 3 (Statements on the measures $\mu^{Z;(T,p)}$)**:** For (T,p) in $\mathbb{R}_+^* \times \left[2, \frac{2}{d-2}\right)$ when $d > 2$ or $\mathbb{R}_+^* \times [2, \infty)$ when $d = 2$ or $d = 1$, let $i_{(T,p)}$ denote the continuous injections from $X^{(T,p)}$ into $\mathrm{L}^2(0,T;\mathrm{L}^2)$ and $\mu^{Z;L} = (i_{(T,p)})_* \mu^{Z;(T,p)}$. The $\sigma-$field on $\mathrm{L}^2(0,T;\mathrm{L}^2)$ is the Borel $\sigma-$field. Let $h \in \mathrm{L}^2(0,T;L^2)$, then

$$\left(h, i_{(T,p)}(Z)\right)_{\mathrm{L}^2(0,T;\mathrm{L}^2)} = \int_0^T \sum_{i,j=1}^{\infty} \int_0^t (e_j, S(t-s)\Phi e_i)_{\mathrm{L}^2} d\beta_i(s) (h(t), e_j)_{\mathrm{L}^2}$$

and from classical computation it is the almost sure limit of a sum of independent centered Gaussian random variables, thus $\mu^{Z;L}$ is a centered Gaussian measure.

Every linear continuous functional on $\mathrm{L}^2(0,T;\mathrm{L}^2)$ defines by restriction a linear continuous functional on $X^{(T,p)}$. Thus, $\mathrm{L}^2(0,T;\mathrm{L}^2)^*$ could be thought of as a subset of $\left(X^{(T,p)}\right)^*$. Since $i_{(T,p)}$ is a continuous injection, $\mathrm{L}^2(0,T;\mathrm{L}^2)^*$ is dense in $\left(X^{(T,p)}\right)^*$ for the weak* topology $\sigma\left(\left(X^{(T,p)}\right)^*, X^{(T,p)}\right)$. This means that, given $x^* \in \left(X^{(T,p)}\right)^*$, there exists a sequence $(h_n)_{n\in\mathbb{N}}$ of elements of $\mathrm{L}^2(0,T;\mathrm{L}^2)$ such that for every $x \in X^{(T,p)}$,

$$\lim_{n\to\infty} \left(h_n, i_{(T,p)}(x)\right)_{\mathrm{L}^2(0,T;\mathrm{L}^2)} = < x^*, x >_{\left(X^{(T,p)}\right)^*, X^{(T,p)}} .$$

In other words, the random variable $< x^*, \cdot >_{\left(X^{(T,p)}\right)^*, X^{(T,p)}}$ is a pointwise limit of $\left(h_n, i_{(T,p)}(\cdot)\right)_{\mathrm{L}^2(0,T;\mathrm{L}^2)}$ which are, from the above, centered Gaussian random variables. As a consequence, $\mu^{Z;(T,p)}$ is a centered Gaussian measure.

Recall that the RKHS $H_{\mu^{Z;L}}$ of $\mu^{Z;L}$ is $\mathrm{Im}R^L$ where R^L is the mapping from $H^*_{\mu^{Z;L}} = \overline{\mathrm{L}^2(0,T;\mathrm{L}^2)^*}^{\mathrm{L}^2(\mu^{Z;L})}$ with the inner product derived from the one in $\mathrm{L}^2(\mu^{Z;L})$ into $\mathrm{L}^2(0,T;\mathrm{L}^2)$ defined for φ in $H^*_{\mu^{Z,L}}$ by

$$R^L(\varphi) = \int_{\mathrm{L}^2(0,T;\mathrm{L}^2)} x\varphi(x)\mu^{Z;L}(dx).$$

The same is true for $H_{\mu^{Z;(T,p)}}$ replacing $\mathrm{L}^2(0,T;\mathrm{L}^2)$ by $X^{(T,p)}$ and $\mu^{Z;L}$ by $\mu^{Z;(T,p)}$.

Since $\mu^{Z;L}$ is the image of $\mu^{Z;(T,p)}$, taking $x^* \in \mathrm{L}^2(0,T;\mathrm{L}^2)^*$, we obtain that

$$\begin{aligned} \|x^*\|_{\mathrm{L}^2(\mu^{Z;L})} &= \textstyle\int_{\mathrm{L}^2(0,T;\mathrm{L}^2)} < x^*, x >^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)^*,\mathrm{L}^2(0,T;\mathrm{L}^2)} \mu^{Z;L}(dx) \\ &= \textstyle\int_{X^{(T,p)}} < x^*, x >^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)^*,\mathrm{L}^2(0,T;\mathrm{L}^2)} \mu^{Z;(T,p)}(dx) \\ &= \textstyle\int_{X^{(T,p)}} < x^*, x >^2_{\left(X^{(T,p)}\right)^*,X^{(T,p)}} \mu^{Z;(T,p)}(dx) = \|x^*\|_{\mathrm{L}^2(\mu^{Z;(T,p)})}. \end{aligned}$$

Therefore, from Lebesgue's dominated convergence theorem, we obtain that

$$\left(X^{(T,p)}\right)^* = \overline{\mathrm{L}^2(0,T;\mathrm{L}^2)^*}^{\sigma\left(\left(X^{(T,p)}\right)^*,X^{(T,p)}\right)} \subset \overline{\mathrm{L}^2(0,T;\mathrm{L}^2)^*}^{\mathrm{L}^2(\mu^{Z;(T,p)})} = H^*_{\mu^{Z;L}}.$$

It follows that $H^*_{\mu^{Z;(T,p)}} \subset H^*_{\mu^{Z;L}}$.

The reverse inclusion follows from the fact that $\mathrm{L}^2(0,T;\mathrm{L}^2)^* \subset \left(X^{(T,p)}\right)^*$.

The conclusion follows from the quite standard fact that the RKHS of $\mu^{Z;L}$, which is a centered Gaussian measure on a Hilbert space, is equal to $\mathrm{Im}\mathcal{Q}^{\frac{1}{2}}$, with the norm of the image structure. $\mathcal{Q}$ denotes the covariance operator of the centered Gaussian measure, it is given, see [34], for $h \in \mathrm{L}^2(0,T;\mathrm{L}^2)$, by

$$\mathcal{Q}h(v) = \int_0^T \int_0^{u\wedge v} IS(v-s)\Phi\Phi^* S(s-u)I^* h(u)dsdu.$$

Corollary B.5 of reference [34] finally gives that $\mathrm{Im}\mathcal{L} = \mathrm{Im}\mathcal{Q}^{\frac{1}{2}}$. □

A.2.5 Continuity with respect to the perturbation

Recall that the mild solution of stochastic NLS (A.1.3) could be written as a function of the perturbation.

Let $v(x)$ denotes the solution of

$$\left\{\begin{array}{l} i\frac{\mathrm{d}v}{\mathrm{d}t} - (\Delta v + |v - ix|^{2\sigma}(v - ix)) = 0, \\ v(0) = u_0, \end{array}\right. \tag{A.2.2}$$

or equivalently a fixed point of the functional

$$\mathcal{F}_z(v)(t) = S(t)u_0 - i\lambda \int_0^t S(t-s)(|(v-iz)(s)|^{2\sigma}(v-iz)(s))ds,$$

where z is an element of $X^{(T,p)}$, p is such that $p \geq 2\sigma + 2$ and (T,p) is an arbitrary pair in $\mathbb{R}_+^* \times \left[2, \frac{2}{d-2}\right)$ when $d > 2$ or $\mathbb{R}_+^* \times [2, \infty)$ when $d = 2$ or $d = 1$.

If u is such that $u = v(Z) - iZ$ where Z is the stochastic convolution, note that its regularity is given in the previous section, then u is a solution of (A.1.3). Consequently, if $\mathcal{G}$ denotes the mapping that satisfies $\mathcal{G}(z) = v(z) - iz$ we obtain that $u = \mathcal{G}(Z)$.

The local existence follows from the fact that for $R > 0$ and $r > 0$ fixed, taking $\|z\|_{X^{(T,2\sigma+2)}} \leq R$ and $\|u_0\|_{\mathrm{H}^1} \leq r$, there exists a sufficiently small $T^*_{2\sigma+2}$ such that the closed ball centered at 0 of radius $2r$ is invariant and $\mathcal{F}_z$ is a contraction for the topology of $\mathrm{L}^\infty([0, T^*_{2\sigma+2}]; \mathrm{L}^2) \cap \mathrm{L}^r(0, T^*_{2\sigma+2}; \mathrm{L}^p)$. Note that a closed ball of $X^{(T^*_{2\sigma+2}, 2\sigma+2)}$ is complete for the topology of $\mathrm{L}^\infty([0, T^*_{2\sigma+2}]; \mathrm{L}^2) \cap \mathrm{L}^r(0, T^*_{2\sigma+2}; \mathrm{L}^p)$. The proof uses extensively the Strichartz' estimates; see [37] for a detailed proof. The same fixed point argument can be used for $\|z\|_{X^{(T,p)}} \leq R$ in a closed ball of radius $2r$ in $X^{(T^*_p, p)}$ for every T^*_p sufficiently small and $p \geq 2\sigma + 2$ such that $(T^*_p, p) \in J$. From (A.2.1), there exists a unique maximal solution $v(z)$ that belongs to $\mathcal{E}_\infty$.

It could be deduced from Proposition 3.5 of [37], that the mapping $\mathcal{G}$ from $\mathcal{X}_\infty$ into $\mathcal{E}_\infty$ is a continuous mapping from $\bigcap_{T \in \mathbb{R}_+^*} X^{(T,2\sigma+2)}$ with the projective limit topology into $\mathcal{E}(\mathrm{H}^1)$. The result can be strengthened as follows.

Proposition A.2.3 *The mapping $\mathcal{G}$ from $\mathcal{X}_\infty$ into $\mathcal{E}_\infty$ is continuous.*

Proof. Let $\tilde{z}$ be a function of $\mathcal{X}_\infty$ and $T < \mathcal{T}(\tilde{Z})$. Revisiting the proof of Proposition 3.5 of [37] and taking $\epsilon > 0$, $p' \geq 2\sigma + 2$, $R = 1 + \|\tilde{z}\|_{X^{(T,p')}}$, $r = 1 + \|v(\tilde{z})\|_{\mathrm{C}([0,T];\mathrm{H}^1)}$, and $2 < p < p'$, there exists $\eta > 0$ satisfying $\eta < \frac{\epsilon}{2(d+1)} \wedge 1$ such that

$$\forall z \in \mathcal{X}_\infty : \ \|z - \tilde{z}\|_{X^{(T,p')}} \leq \eta, \ \|v(z) - v(\tilde{z})\|_{\mathrm{C}([0,T];\mathrm{H}^1)} \leq \left(\frac{\epsilon}{2(d+1)(4r)^\alpha}\right)^{\frac{1}{1-\alpha}} \wedge 1.$$

The constant α is the one that appears in the application of Hölder's inequality before (A.2.1). Consequently, since $v(z)$ and $v(\tilde{z})$ are functions of the closed ball centered at 0 and of radius $2r$ in $X^{(T,p)}$, the triangle inequality gives that

$$\|v(z) - v(\tilde{z})\|_{X^{(T,p')}} \leq 4r.$$

The application of both Hölder's inequality and the triangle inequality allow to conclude that

$$\forall z \in \mathcal{X}_\infty : \|z - \tilde{z}\|_{X^{(T,p')}} \leq \eta, \ \|\mathcal{G}(z) - \ddagger(\tilde{x})\|_{X^{(T,p)}} \leq \epsilon$$

which, from the definition of the neighborhood basis of $\mathcal{E}_\infty$, gives the continuity. □

The following corollary is a consequence of the last statement of Section A.2.2.

Corollary A.2.4 *In the focusing subcritical case or in the defocusing case, $\mathcal{G}$ is a continuous mapping from $\mathcal{X}_\infty$ into $\mathcal{X}_\infty$*

The continuity allows us to define the law of the solutions of the stochastic NLS equations on $\mathcal{E}_\infty$ and in the cases of global existence in $\mathcal{X}_\infty$ as the direct image $\mu^u = \mathcal{G}_*\mu^Z$, the same notation will be used in both cases.

Let consider the solutions of

$$i\mathrm{d}u^\epsilon - (\Delta u^\epsilon + \lambda|u^\epsilon|^{2\sigma}u^\epsilon)\mathrm{d}t = \sqrt{\epsilon}\mathrm{d}W, \tag{A.2.3}$$

where $\epsilon \geq 0$. The laws of the solutions u^ϵ in the corresponding trajectory spaces are denoted by μ^{u^ϵ}, or equivalently $\mathcal{G}_*\mu^{Z_\epsilon}$ where μ^{Z_ϵ} is the direct image of μ^Z under the transformation $x \mapsto \sqrt{\epsilon}x$ on $\mathcal{X}_\infty$. The continuity also gives that the family converges weakly to the Dirac mass on the deterministic solution u_d as ϵ converges to zero. Next section is devoted to the study of the convergence towards 0 of rare events or tail events of the law of the solution u^ϵ, namely large deviations. It allows to describe more precisely the convergence towards the deterministic measure.

A.3 Sample path large deviations

Theorem A.3.1 *The family of probability measures $\left(\mu^{u^\epsilon}\right)_{\epsilon\geq 0}$ on $\mathcal{E}_\infty$ satisfies a LDP of speed ϵ and good rate function*

$$I(u) = \frac{1}{2}\inf_{h\in \mathrm{L}^2(0,\infty;\mathrm{L}^2):\mathbf{S}(h)=u}\left\{\|h\|^2_{\mathrm{L}^2(0,\infty;\mathrm{L}^2)}\right\},$$

where $\inf \emptyset = \infty$ *and* $\mathbf{S}(h)$*, called the skeleton, is the unique mild solution of the following control problem:*

$$\left\{ \begin{array}{l} i\frac{\mathrm{d}u}{\mathrm{d}t} = \Delta u + \lambda |u|^{2\sigma} u + \Phi h, \\ u(0) = u_0 \ \in \mathrm{H}^1. \end{array} \right.$$

This is to say that for every Borel set A *of* $\mathcal{E}_\infty$*,*

$$-\inf_{u \in \overset{\circ}{A}} I(u) \leq \underline{\lim}_{\epsilon \to 0} \epsilon \log \mu^{u^\epsilon}(A) \leq \overline{\lim}_{\epsilon \to 0} \epsilon \log \mu^{u^\epsilon}(A) \leq -\inf_{u \in \overline{A}} I(u).$$

The same result holds in $\mathcal{X}_\infty$ *for the family of laws of the solutions in the cases of global existence.*

Proof. The general LDP for centered Gaussian measures on real Banach spaces, see [53], gives that for a given pair (T,p) in $\mathbb{R}_+^* \times \left[2, \frac{2}{d-2}\right)$ when $d > 2$ or $\mathbb{R}_+^* \times [2, \infty)$ when $d = 2$ or $d = 1$, the family $\left(p_{(T,p)*}\mu^{Z_\epsilon}\right)_{\epsilon \geq 0}$ satisfies a LDP on $X^{(T,p)}$ of speed ϵ and good rate function defined for $z \in X^{(T,p)}$ by,

$$I^{Z;(T,p)}(z) = \left\{ \begin{array}{ll} \frac{1}{2}\|z\|^2_{H_{\mu^{Z;(T,p)}}}, & \text{if } z \in H_{\mu^{Z;(T,p)}}, \\ \infty, & \text{otherwise,} \end{array} \right.$$

which, using Proposition A.2.2, is equal to

$$I^{Z;(T,p)}(z) = \left\{ \begin{array}{ll} \frac{1}{2}\|z|^2_{\mathrm{Im}\mathcal{L}}, & \text{if } z \in \mathrm{Im}\mathcal{L}, \\ \infty, & \text{otherwise,} \end{array} \right. .$$

Dawson-Gärtner's theorem, see [48], along with the monotone convergence theorem, allows us to deduce that the family $\left(\mu^{Z_\epsilon}\right)_{\epsilon \geq 0}$ satisfies the LDP with the good rate function defined for $z \in \mathcal{X}_\infty$ by

$$\begin{array}{ll} I^Z(z) & = \sup_{(T,p) \in J} \left\{ I^{Z;(T,p)}(z) \right\} \\ & = \left\{ \begin{array}{l} \sup_{(T,p) \in J} \left\{ \frac{1}{2} \| \left(\Phi_{|\mathrm{Ker}\Phi^\perp} \right)^{-1} \left(\frac{\mathrm{d}z}{\mathrm{d}t} + i\Delta z \right) \|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \right\} \\ \infty \quad \text{if } \frac{\mathrm{d}z}{\mathrm{d}t} + i\Delta z \notin \mathrm{Im}\Phi \end{array} \right. \\ & = \frac{1}{2} \inf_{h \in \mathrm{L}^2(0,\infty;\mathrm{L}^2) : \mathcal{L}(h) = z} \left\{ \|h\|^2_{\mathrm{L}^2(0,\infty;\mathrm{L}^2)} \right\}. \end{array}$$

It has been shown in Section A.2.2 and A.2.5 that $\mathcal{G}$ is a continuous function from a Hausdorff topological space into another Hausdorff topological space. Consequently, both results follow from Varadhan's contraction principle along with the fact that if $\mathcal{G} \circ \mathcal{L}(h) = u$ then u is the unique mild solution of the control problem (i.e. $u = \mathbf{S}(h)$). □

Remark A.3.2 *The rate function can be written*

$$I(u) = \begin{cases} \frac{1}{2}\int_0^{\mathcal{T}(u)} \left\| \left(\Phi_{|\mathrm{Ker}\Phi^\perp}\right)^{-1} \left(i\frac{\mathrm{d}u}{\mathrm{d}t} - \Delta u - \lambda |u|^{2\sigma}u\right)(s) \right\|_{\mathrm{L}^2} ds \\ \text{if } i\frac{\mathrm{d}u}{\mathrm{d}t} - \Delta u - \lambda|u|^{2\sigma}u \in \mathrm{Im}\Phi, \\ \infty \qquad \text{otherwise.} \end{cases}$$

Remark A.3.3 *In the cases where blow-up may occur, the argument that will follow allows us to prove the weaker result that, given an* (T,p) *in* $\mathbb{R}_+^* \times \left[2, \frac{2}{d-2}\right)$ *when* $d > 2$ *or* $\mathbb{R}_+^* \times [2,\infty)$ *when* $d = 2$ *or* $d = 1$ *and*

$$I^{(T,p)}(u) = \frac{1}{2} \inf_{h \in \mathrm{L}^2(0,T;\mathrm{L}^2): \mathbf{S}(h)=u} \left\{ \|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \right\},$$

then for every bounded Borel set A *of* $X^{(T,p)}$

$$-\inf_{u \in \overset{\circ}{A}} I^{(T,p)}(u) \le \underline{\lim}_{\epsilon\to 0} \epsilon \log \mathbb{P}(u^\epsilon \in A) \le \overline{\lim}_{\epsilon \to 0} \epsilon \log \mathbb{P}(u^\epsilon \in A) \le -\inf_{u \in \overline{A}} I^{(T,p)}(u). \tag{A.3.1}$$

Indeed, if u^ϵ belongs to A, there exists a constant R such that $\|u^\epsilon\|_{X^{(T,p)}} \le R$. Denoting by $u^{\epsilon,R}$ the solution of the following fixed point problem

$$u^{\epsilon,R}(t) = S(t)u_0 - i\lambda \int_0^t S(t-s)(|(u^{\epsilon,R} - i\sqrt{\epsilon}Z)(s)|^{2\sigma}(u^{\epsilon,R} - i\sqrt{\epsilon}Z)(s))\mathbb{1}_{\|u^{\epsilon,R}\|_{X^{(s,p)}} \le R} ds,$$

the arguments used previously allow to show that $\sqrt{\epsilon}Z \to u^{\epsilon,R}$ is a continuous mapping from every $X^{(T,p')}$ into $X^{(T,p)}$ for $p' > p$. The result (A.3.1) with $u^{\epsilon,R}$ follows from Varadhan's contraction principle replacing $\mathbf{S}(h)$ by $\mathbf{S}^R(h)$ with the truncation in front of the nonlinearity. Finally, the statement follows from the fact that $\|u^\epsilon\|_{X^{(T,p)}} \le R$ implies that $u^{\epsilon,R} = u^\epsilon$ and that

$$\inf_{h \in \mathrm{L}^2(0,T;\mathrm{L}^2): \mathbf{S}^R(h) \in \overline{A}} \{\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)}\} = \inf_{h \in \mathrm{L}^2(0,T;\mathrm{L}^2): \mathbf{S}(h) \in \overline{A}} \{\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)}\}.$$

Note that writing $\frac{\partial}{\partial t}h$ instead of h in the optimal control problem leads to a rate function consisting in the minimisation of $\frac{1}{2}\|h\|^2_{H_0^1(0,\infty;\mathrm{L}^2)}$. This space is somehow the equivalent of the Cameron-Martin space for the Brownian motion. Specifying only the law μ of $W(1)$ on H^1 and dropping Φ in the control problem would lead to a rate function consisting in the minimisation of $\frac{1}{2}\|h\|^2_{H_0^1(0,\infty;H_\mu)}$, where H_μ stands for the RKHS of μ.

The formalism of a LDP stated in the intersection space with a projective limit topology allows, for example, to deduce by contraction, when there is no blow-up in finite time, a variety of sample path LDP on every $X^{(T,p)}$. The rate function could be interpreted as the minimal energy to implement control.

LDP for the family of laws of $u^\epsilon(T)$, for a fixed T, could be deduced by contraction in the cases of global existence. The rate function is then the minimal energy needed to transfer u_0 to x from 0 to T.

Next section gives a characterization of the support of the law of the solution in our setting. Section A.5 is devoted to some consequences of these results on the blow-up time.

A.4 The support of the law of the solution

Theorem A.4.1 (The support theorem) *The support of the law of the solution is characterized by*

$$supp\ \mu^u = \overline{\mathrm{Im}\mathbf{S}}^{\mathcal{E}_\infty}$$

and in the cases of global existence by

$$supp\ \mu^u = \overline{\mathrm{Im}\mathbf{S}}^{\mathcal{X}_\infty}$$

Proof. Step 1: From Proposition A.2.3, given (T,p) in $\mathbb{R}_+^* \times \left[2, \frac{2}{d-2}\right)$ when $d>2$ or $\mathbb{R}_+^* \times [2,\infty)$ when $d=2$ or $d=1$, $\mu^{Z;(T,p)}$ is a Gaussian measure on a Banach space and its RKHS is $\mathrm{Im}\mathcal{L}$. Consequently, see [8] Theorem (IX,2;1), its support is $\overline{\mathrm{Im}\mathcal{L}}^{X^{(T,p)}}$. Also, from the definition of the image measure we have that

$$\mu^Z\left(p_{(T,p)}^{-1}\left(\overline{\mathrm{Im}\mathcal{L}}^{X^{(T,p)}}\right)\right) = \mu^{Z;(T,p)}\left(\overline{\mathrm{Im}\mathcal{L}}^{X^{(T,p)}}\right) = 1.$$

As a consequence the first inclusion follows

$$supp\ \mu^Z \subset \bigcap_{(T,p)} p_{(T,p)}^{-1}\left(\overline{\mathrm{Im}\mathcal{L}}^{X^{(T,p)}}\right) = \overline{\mathrm{Im}\mathcal{L}}^{\mathcal{X}_\infty}.$$

It then suffices to show that $\mathrm{Im}\mathcal{L} \subset supp\ \mu^Z$. Suppose that $x \notin supp\ \mu^Z$, then there exists a neighborhood V of x in $\mathcal{X}_\infty$, satisfying $V = \bigcap_{i=1}^n V^{(T_i,p_i)}$ where $V^{(T_i,p_i)}$ is a neighborhood of x in $X^{(T_i,p_i)}$, n is a finite integer and

(T_i, p_i) a finite sequence of elements of $\mathbb{R}_+^* \times \left[2, \frac{2}{d-2}\right)$ when $d > 2$ or $\mathbb{R}_+^* \times [2, \infty)$ when $d = 2$ or $d = 1$, such that $\mu^Z(V) = 0$. It can be shown that $\bigcap_{i=1}^n X^{(T_i,p_i)}$ is still a separable Banach space. It is such that $\mathcal{X}_\infty$ is continuously embedded into it, and such that the Borel direct image probability measure is a Gaussian measure of RKHS $\mathrm{Im}\mathcal{L}$. The support of this measure is then the closure of $\mathrm{Im}\mathcal{L}$ for the topology defined by the maximum of the norms on each factor. Thus, $V \cap \mathrm{Im}\mathcal{L} = \emptyset$ and $x \notin \mathrm{Im}\mathcal{L}$.

Step 2: We conclude using the continuity of $\mathcal{G}$.

Indeed since $\mathcal{G}(\mathrm{Im}\mathcal{L}) \subset \overline{\mathcal{G}(\mathrm{Im}\mathcal{L})}^{\mathcal{E}_\infty}$, $\mathrm{Im}\mathcal{L} \subset \mathcal{G}^{-1}\left(\overline{\mathcal{G}(\mathrm{Im}\mathcal{L})}^{\mathcal{E}_\infty}\right)$. Since $\mathcal{G}$ is continuous, the right side is a closed set of $\mathcal{X}_\infty$ and from step 1,

$$supp\ \mu^Z \subset \mathcal{G}^{-1}\left(\overline{\mathrm{Im}\,(\mathcal{G} \circ \mathcal{L})}^{\mathcal{E}_\infty}\right),$$

and

$$\mu^Z\left(\mathcal{G}^{-1}\left(\overline{\mathrm{Im}\mathbf{S}}^{\mathcal{E}_\infty}\right)\right) = 1,$$

thus

$$supp\ \mu^u \subset \overline{\mathrm{Im}\mathbf{S}}^{\mathcal{E}_\infty}.$$

Suppose that $x \notin supp\ \mu^u$, there exists a neighborhood V of x in $\mathcal{E}_\infty$ such that $\mu^u(V) = \mu^Z\left(\mathcal{G}^{-1}(V)\right) = 0$, consequently $\mathcal{G}^{-1}(V) \bigcap \mathrm{Im}\mathcal{L} = \emptyset$ and $x \notin \mathrm{Im}\mathbf{S}$. This gives the reverse inclusion.

The same arguments hold replacing $\mathcal{E}_\infty$ by $\mathcal{X}_\infty$. □

Note that the result of step 2 is general and gives that the support of the direct images μ^E of the law μ^u by any continuous mapping f from either $\mathcal{E}_\infty$ or $\mathcal{X}_\infty$ into a topological vector space E is $\overline{\mathrm{Im}\,(f \circ \mathbf{S})}^E$. For example, in the cases of global existence, given a positive T, the support of the law in H^1 of $u(T)$ is $\overline{\mathrm{Im}\mathbf{S}(T)}^{\mathrm{H}^1}$.

Remark A.4.2 *Remark that the LDP and support theorem may be proved for more general driving noises provided that the stochastic convolution remains a Gaussian process. The case of a noise derived from a fractional Wiener process which is a one parameter generalization of the usual Wiener process has been studied. The results will appear elsewhere.*

A.5 Applications to the blow-up times

In this section the equation with a focusing nonlinearity, i.e. $\lambda = 1$, is considered. In this case, it is known that some solutions of the deterministic equation blow up in finite time for a critical or subcritical nonlinearity. It has been proved in Section A.2.2 that $\mathcal{T}$ is a measurable mapping from $\mathcal{E}_\infty$ to $[0,\infty]$, both spaces are equipped with their Borel $\sigma-$fields. Incidentally, $\mathcal{T}(u)$ is a $\mathcal{F}_t-$stopping time. Also, if B is a Borel set of $[0,\infty]$, $\mathbb{P}\left(\mathcal{T}(u)\in B\right)=\mu^u\left(\mathcal{T}^{-1}(B)\right)$.

The support theorem allows us to determine whether an open or a closed set of the form $\mathcal{T}^{-1}(B)$ is such that $\mu^u\left(\mathcal{T}^{-1}(B)\right)>0$ or $\mu^u\left(\mathcal{T}^{-1}(B)\right)<1$ respectively. An application of this fact is given in Proposition A.5.1. For a Borel set B such that $\left\{Int\left(\mathcal{T}^{-1}(B)\right)\right\}\cap\overline{\mathrm{Im}\mathbf{S}}^{\mathcal{E}_\infty}$ is nonempty, where $Int\left(\mathcal{T}^{-1}(B)\right)$ stands for the interior set of $\mathcal{T}^{-1}(B)$, $\mathbb{P}(\mathcal{T}(u)\in B)>0$ holds.

Also, $\mathcal{T}$ is not continuous and Varadhan's contraction principle does not allow to obtain a LDP for the law of the blow-up time. Nonetheless, the LDP for the family $\left(\mu^{u^\epsilon}\right)_{\epsilon>0}$ gives the interesting result that

$$\left\{\begin{array}{l} -\inf_{u\in Int(\mathcal{T}^{-1}(B))} I(u)\leq \underline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(\mathcal{T}(u^\epsilon)\in B\right)\\ \overline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(\mathcal{T}(u^\epsilon)\in B\right)\leq -\inf_{u\in\overline{\mathcal{T}^{-1}(B)}} I(u).\end{array}\right.$$

Note also that the interior or the closure of sets in $\mathcal{E}_\infty$ are difficult to characterize. In that respect, the semicontinuity of $\mathcal{T}$ makes the sets $(T,\infty]$ and $[0,T]$ particularly interesting.

A.5.1 Probability of blow-up after time T

Proposition A.5.1 *If $u_0\in\mathrm{H}^3$ and the range of Φ is dense then for every positive T,*

$$\mathbb{P}(\mathcal{T}(u)>T)>0.$$

Proof. Since $\mathcal{T}$ is lower semicontinuous, $\mathcal{T}^{-1}((T,\infty])$ is an open set.

Consider $H=-\Delta u_0-|u_0|^{2\sigma}u_0$ which satisfies $\mathcal{G}\circ\Lambda(H)=u_0$, where Λ has been defined in Section A.2.1, then $\mathcal{T}(\mathbf{S}(H))=\infty$. Also, using Φ one defines, in a natural way, an operator from $\mathrm{L}^2_{loc}(0,\infty;\mathrm{L}^2)$ into $\mathrm{L}^2_{loc}(0,\infty;\mathrm{H}^1)$ and it can be shown, that it still has a dense range. Consequently, there exists a sequence $(h_n)_{n\in\mathbb{N}}$ of $\mathrm{L}^2_{loc}(0,\infty;\mathrm{L}^2)$ functions such that $(\Phi(h_n))_{n\in\mathbb{N}}$ converges to H in $\mathrm{L}^2_{loc}(0,\infty;\mathrm{H}^1)$.

Using the semicontinuity of $\mathcal{T}$, the continuity of $\mathcal{G}$, the fact that $\mathbf{S}=\mathcal{G}\circ\Lambda\circ\Phi$, the following lemma and the fact that $\mathrm{L}^2_{loc}(0,\infty;\mathrm{H}^1)$ is continuously embedded in $\mathrm{L}^1_{loc}(0,\infty;\mathrm{H}^1)$, $\underline{\lim}_{n\to\infty}\mathcal{T}(\mathbf{S}(h_n))\geq\infty$, i.e. $\lim_{n\to\infty}\mathcal{T}(\mathbf{S}(h_n))=\infty$, follows. Therefore $\mathcal{T}(\mathbf{S}(h_n))>T$ for n large enough and $\mathcal{T}^{-1}((T,\infty])\cap(\mathrm{Im}\mathbf{S})$ is nonempty.

The conclusion follows then from the support theorem. $\square$

As a corollary, taking the complement of $\mathcal{T}^{-1}((T,\infty])$, $\mathbb{P}(\mathcal{T}(u) \leq T) < 1$ follows. This is related to the results of [38] where it is proved that for every positive T, $\mathbb{P}(\mathcal{T}(u) < T) > 0$ and to the graphs in Section A.4 of [43].

Lemma A.5.2 *The operator* Λ*, defined in Section A.2.1, is continuous from* $\mathrm{L}^1_{loc}(0,\infty;\mathrm{H}^1)$ *into* $\mathcal{X}_\infty$*.*

Proof. The result follows from (ii) of the Strichartz inequalities when $s = 1$ and $\rho = 2$, the fact that the partial derivatives with respect to one space variable commutes with both the integral and the group and the definition of the projective limit topology. □

The following result holds when the amplitude of the noise converges to zero.

Proposition A.5.3 *If* $u_0 \in \mathrm{H}^3$*, the range of* Φ *is dense and* $T \geq \mathcal{T}(u_d)$*, where* u_d *is the solution of the deterministic NLS equation with initial datum* u_0*, there exists* c *in* $[0,\infty)$ *such that*

$$\underline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(\mathcal{T}(u^\epsilon) > T\right) \geq -c.$$

Proof. Define

$$L^{(T,\infty]} = \frac{1}{2}\inf_{h\in\mathrm{L}^2(0,\infty;\mathrm{L}^2):\mathcal{T}(\mathbf{S}(h))>T}\left\{\|h\|^2_{\mathrm{L}^2(0,\infty;\mathrm{L}^2)}\right\}.$$

The result follows then from

$$-L^{(T,\infty]} \leq \underline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(\mathcal{T}(u^\epsilon) > T\right)$$

and that, from the arguments of the proof of Proposition A.5.1, for every T such that $T \geq \mathcal{T}(u_d)$ the set $\{h \in \mathrm{L}^2(0,\infty;\mathrm{L}^2) : \mathcal{T}(\mathbf{S}(h)) > T\}$ is nonempty. □

Remark A.5.4 *The assumption that* $u_0 \in \mathrm{H}^3$ *could be dropped using similar arguments as in Proposition* 3.3 *of [38].*

Note that the LDP does not give interesting information on the upper bound even if the bounds have been sharpened using the rather strong projective limit topology. It is zero since $h = 0$ belongs to $\overline{\mathcal{T}^{-1}((T,\infty])}$ as for every $T > 0$, $\overline{\mathcal{T}^{-1}((T,\infty])} = \mathcal{E}_\infty$. Indeed, if a function f of $\mathcal{E}_\infty$ is given and blows up at a particular time $\mathcal{T}(f)$ such that $T > \mathcal{T}(f)$, it is possible to build a sequence $(f_n)_{n\in\mathbb{N}}$ of functions of $\mathcal{E}_\infty$ equal to f on $\left[0, \mathcal{T}(f) - \frac{1}{n}\right]$ and such that $\mathcal{T}(f_n) > T$. The same problem will appear in the next section where the LDP gives a lower bound equal to $-\infty$. Indeed, $Int\left(\mathcal{T}^{-1}([0,T])\right)$ is the complement of the above and thus an empty set. To overcome this problem the approximate blow-up time is introduced. Note also that it is possible that $L^{(T,\infty]}=0$.

Also, the case $T<\mathcal{T}(u_d)$ has not been treated. Indeed, the associated event is not a large deviation event and the LDP only gives that

$$\lim_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(\mathcal{T}(u^\epsilon)>T\right)=0.$$

A.5.2 Probability of blow-up before time T

In that case we obtain

$$-\infty\leq\underline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(\mathcal{T}(u^\epsilon)\leq T\right)\leq\overline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(\mathcal{T}(u^\epsilon)\leq T\right)\leq -U^{[0,T]}$$

where $U^{[0,T]}=\frac{1}{2}\inf_{h\in \mathrm{L}^2(0,\infty;\mathrm{L}^2):\mathcal{T}(\mathbf{S}(h))\leq T}\left\{\|h\|^2_{\mathrm{L}^2(0,\infty;\mathrm{L}^2)}\right\}$.

Proposition A.5.5 *If $T<\mathcal{T}(u_d)$,*

$$\overline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(\mathcal{T}(u^\epsilon)\leq T\right)\leq -U^{[0,T]}<0.$$

Proof. Let $(h_n)_{n\in\mathbb{N}}$ be a sequence of $\mathrm{L}^2(0,\infty;\mathrm{L}^2)$ functions converging to zero. It follows from Lemma 5.2 and the fact that $\mathrm{L}^2(0,\infty;\mathrm{H}^1)$ is continuously embedded into $\mathrm{L}^1_{loc}(0,\infty;\mathrm{H}^1)$ that $\mathbf{S}=\mathcal{G}\circ\Lambda\circ\Phi$ is continuous from $\mathrm{L}^2(0,\infty;\mathrm{L}^2)$ into $\mathcal{E}_\infty$. Also, from the semicontinuity of $\mathcal{T}$, $\underline{\lim}_{n\to\infty}\mathcal{T}(\mathbf{S}(h_n))\geq\mathcal{T}(u_d)$. Thus there exists N large enough such that for every $n\geq N$, $\mathcal{T}(\mathbf{S}(h_n))>T$. As a consequence we obtain that necessarily $U^{[0,T]}>0$. $\square$

When $T\geq\mathcal{T}(u_d)$, the probability is not supposed to tend to zero. Also, as $h=0$ is a solution, the upper bound is zero and none of the bounds are interesting.

A.5.3 Bounds for the approximate blow-up time

To overcome the limitation that $\overline{\mathcal{T}^{-1}((T,\infty])}=\mathcal{E}_\infty$, which does not allow to have two interesting bounds simultaneously, we introduce for every positive R the mappings $\mathcal{T}_R$ defined for $f\in\mathcal{E}_\infty$ by

$$\mathcal{T}_R(f)=\inf\{t\in[0,\infty):\|f(t)\|_{\mathrm{H}^1}\geq R\}.$$

It corresponds to the approximation of the blow-up time used in [43]. We obtain the following bounds.

Proposition A.5.6 *When $T \geq \mathcal{T}_R(u_d)$, the following inequality holds*

$$\begin{array}{l} -c < -L_R^{(T,\infty]} \leq \underline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(\mathcal{T}_R(u^\epsilon) > T\right) \\ \leq \overline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(\mathcal{T}_R(u^\epsilon) > T\right) \leq -\sup_{\alpha>0} L_{R+\alpha}^{(T,\infty]}. \end{array}$$

Also, when $T < \mathcal{T}_R(u_d)$, we have that

$$\begin{array}{l} -\inf_{\alpha>0} U_{R+\alpha}^{[0,T]} \leq \underline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(\mathcal{T}_R(u^\epsilon) \leq T\right) \\ \leq \overline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(\mathcal{T}_R(u^\epsilon) \leq T\right) \leq -U_R^{[0,T]} < 0. \end{array}$$

In the above c is nonnegative and the numbers $L_R^{(T,\infty]}$ and $U_R^{[0,T]}$ are defined as $L^{(T,\infty]}$ and $U^{[0,T]}$ replacing $\mathcal{T}$ by $\mathcal{T}_R$.

Proof. The result follows from the facts that $\mathcal{T}_R$, which is not continuous, is lower semicontinuous, that for every $\alpha > 0$, $\overline{\mathcal{T}_R^{-1}((T,\infty])} \subset \mathcal{T}_{R+\alpha}^{-1}((T,\infty])$, thus $\mathcal{T}_{R+\alpha}^{-1}([0,T]) \subset Int\left(\mathcal{T}_R^{-1}([0,T])\right)$ and from the arguments used in the proofs of the last two propositions. □

We also obtain the following estimates of other large deviation events.

Corollary A.5.7 *If $S, T < \mathcal{T}_R(u_d)$, for every $c > 0$, there exists $\epsilon_0 > 0$ such that if $\epsilon \leq \epsilon_0$,*

$$\begin{array}{l} \exp\left(-\frac{\inf_{\alpha>0} U_{R+\alpha}^{[0,T]}+c}{\epsilon}\right)\left(1-\exp\left(-\frac{U_R^{[0,S]}-\inf_{\alpha>0} U_{R+\alpha}^{[0,T]}}{\epsilon}\right)\right) \\ \leq \mathbb{P}\left(S < \mathcal{T}_R(u^\epsilon) \leq T\right) \\ \leq \exp\left(-\frac{U_R^{[0,T]}-c}{\epsilon}\right)\left(1-\exp\left(-\frac{\inf_{\alpha>0} U_{R+\alpha}^{[0,S]}-U_R^{[0,T]}}{\epsilon}\right)\right). \end{array}$$

If $S, T > \mathcal{T}_R(u_d)$, for every positive c, there exists a positive ϵ_0 such that if $\epsilon \leq \epsilon_0$,

$$\begin{array}{l} \exp\left(-\frac{L_R^{(S,\infty]}+c}{\epsilon}\right)\left(1-\exp\left(-\frac{\sup_{\alpha>0} L_{R+\alpha}^{(T,\infty]}-L_R^{(T,\infty]}}{\epsilon}\right)\right) \\ \leq \mathbb{P}\left(S < \mathcal{T}_R(u^\epsilon) \leq T\right) \\ \leq \exp\left(-\frac{\sup_{\alpha>0} L_{R+\alpha}^{(S,\infty]}-c}{\epsilon}\right)\left(1-\exp\left(-\frac{L_R^{(T,\infty]}-\sup_{\alpha>0} L_{R+\alpha}^{(S,\infty]}}{\epsilon}\right)\right). \end{array}$$

Proof. When $S, T < \mathcal{T}_R(u_d)$, the result follows from the inequalities and from the fact that

$$\begin{array}{rl} \mathbb{P}\left(S < \mathcal{T}_R(u^\epsilon) \leq T\right) & = \mathbb{P}\left(\{\mathcal{T}_R(u^\epsilon) \leq T\} \setminus \{\mathcal{T}_R(u^\epsilon) \leq S\}\right) \\ & = \mathbb{P}\left(\mathcal{T}_R(u^\epsilon) \leq T\right)\left(1-\frac{\mathbb{P}(\mathcal{T}_R(u^\epsilon)\leq S)}{\mathbb{P}(\mathcal{T}_R(u^\epsilon)\leq T)}\right). \end{array}$$

When $S, T > \mathcal{T}_R(u_d)$, we use

$$\begin{aligned}\mathbb{P}\left(S < \mathcal{T}_R(u^\epsilon) \leq T\right) &= \mathbb{P}\left(\{\mathcal{T}_R(u^\epsilon) > S\} \setminus \{\mathcal{T}_R(u^\epsilon) > T\}\right) \\ &= \mathbb{P}\left(\mathcal{T}_R(u^\epsilon) > S\right)\left(1 - \frac{\mathbb{P}(\mathcal{T}_R(u^\epsilon) > T)}{\mathbb{P}(\mathcal{T}_R(u^\epsilon) > S)}\right).\end{aligned}$$

□

A.6 Applications to nonlinear optics

The NLS equation when $d = \lambda = \sigma = 1$ is called the noisy cubic focusing nonlinear Schrödinger equation. It is a model used in nonlinear optics. Recall that for the above values of the parameters the solutions are global. The variable t stands for the one dimensional space coordinate and x for the time. The deterministic equation is such that there exists a particular class of solutions, which are localized in space (here time), that propagate at a finite constant velocity and keep the same shape. These solutions are called solitons or solitary waves. The functions

$$\Psi_\eta(t, x) = \sqrt{2}\eta \exp\left(-i\eta^2 t\right) \operatorname{sech}(\eta x), \ \ \eta > 0,$$

form a family of solitons. They are used in optical fibers as information carriers to transmit the datum 0 or 1 at high bit rates over long distances. The noise stands for the noise produced by in-line amplifiers.

Let u^ϵ denotes the solution with $u_0(\cdot) = \Psi_1(0, \cdot)$ as initial datum and ϵ as noise amplitude like in Section A.3 and $u^{\epsilon,n}$ denotes the solution with null initial datum and the same noise amplitude. The mass of u_0 is 4.

At a particular coordinate T of the fiber, when a window $[-l, l]$ is given, the square of the $\mathrm{L}^2(-l, l)-$norm, or measured mass, is recorded. It is close to the mass in the deterministic case for sufficiently high l since the wave is localized. A decision criterium is to accept that we have 1 if the measured mass is above a certain threshold and 0 otherwise. We set a threshold of the form $4(1-\gamma)$, where γ is a real number in $[0, 1]$.

As the soliton is progressively distorted by the noise, it is possible either to wrongly decide that the source has emitted a 1, or to wrongly discard a 1. The two error probabilities consist of

$$\mathbb{P}_\epsilon^{|0} = \mathbb{P}\left(\int_{-l}^{l} |u^{\epsilon,n}(T, x)|^2 dx \geq 4(1-\gamma)\right)$$

and

$$\mathbb{P}_\epsilon^{|1} = \mathbb{P}\left(\int_{-l}^{l} |u^{\epsilon}(T, x)|^2 dx < 4(1-\gamma)\right).$$

In modern communication systems the error rate is less than 10^{-9} which is beyond the scope of statistics, moreover due to the nonlinearity of the system the measured mass does not have a gaussian law. This justifies that we use theoretical arguments to characterize these error probabilities. We show in this section how the LDP applies to this problem.

We obtain similar results, in the case of an unbounded window, as in reference [62] for the first error probability and as [63] for the second error probability. In these articles the heuristic argument of the collective coordinates is used. This is a physical argument which unables to reduce the problem to a finite dimensional system involving modulated parameters. In [62], the authors explain what the leading parameters are and reduce the problem to a three dimensional problem, that the fluctuations of the parameters are described by SDEs where the noises are some spatial integrals of the initial noise and that the averaging over the initial noise is equivalent to averaging over new noises with zero cross correlations. They explain that a decrease in the soliton power $Q_\epsilon = \frac{\mathbf{N}(u^\epsilon(T))^2}{4}$ and a timing jitter $T_\epsilon = \frac{\int_{-\infty}^{+\infty} x|u^\epsilon(T,x)|^2 dx}{\mathbf{N}(u^\epsilon(T))^2}$, which characterizes the shifts in the arrival time of the pulse, are mainly responsible for the loss of the pulse. Thus they write down the probability density function of the joint law of the two processes, using a formalism called the instanton formalism, as a quantity which is the averaging over the noise of the path integral over arbitrary functions for the modulated parameters, taking into account the finite dimensional evolution, of the exponential of an integral over t in $[0, T]$ of the so called effective Lagrangian. Finally they compute the path integral using a saddle-point approximation with boundary conditions and obtain an expression of the probability density function in the small noise asymptotic. Details on the calculation are given in [63]. The overall argument seems very difficult to justify rigorously. In particular, the reduction to a three dimensional problem is obtained by minimizing the Lagrangian on a small space of curves whereas NLS is obtained by minimizing over all paths. Note that they recover analytically the empirical Gordon-Haus effect that the dispersion in timing is much larger than that of the mass. The authors also explain that, for the first error probability, the optimal way to create a large signal is to grow a soliton and obtain a small noise asymptotic expression of the probability density function of the amplitude at the coordinate T of the solution with null initial datum.

In the following we make the assumption that $\mathrm{Im}\Phi$ has a dense range. Indeed, from the arguments used in the proof of Proposition A.5.3, it is needed for controllability issues to guarantee that the infima are not taken over empty sets. Also T is fixed, $\gamma_0 \in \left(0, \frac{1}{2}\right)$ is fixed and the size l of the window is such that

$$\int_{-l}^{l} |u_d(T,x)|^2 dx \wedge \int_{-l}^{l} |\Psi_1(0,x)|^2 dx > 4\left(1 - \frac{\gamma_0}{2}\right).$$

We finally stress that for the events we study in dimension $d = 1$ we could consider a L^2 setting instead of a H^1 setting. However we chose to work in H^1 to keep the coherence of the article. The H^1 setting is necessary when we consider higher dimensions or larger powers of nonlinearities and study events like

$$\left\{\int_{-l}^{l} |u^{\epsilon,n}(T,x)|^2 dx \geq 4 - \gamma, \mathcal{T}(u^{\epsilon,n}) > T\right\}$$

or

$$\left\{\int_{-l}^{l} |u^{\epsilon}(T,x)|^2 dx < 4 - \gamma, \mathcal{T}(u^{\epsilon}) > T\right\}.$$

The LDP proved herein allows to state the following proposition.

Proposition A.6.1 *For every γ in $[\gamma_0, 1-\gamma_0]$ besides an at most countable set of points, the following equivalents for the probabilities of error hold*

$$\begin{array}{l} \lim_{\epsilon\to 0} \epsilon \log \mathbb{P}_{\epsilon}^{|0} = -\frac{1}{2} \inf_{h\in \mathrm{L}^2(0,\infty;\mathrm{L}^2): \int_{-l}^{l} |\tilde{\mathbf{S}}(h)(T,x)|^2 dx \geq 4(1-\gamma)} \left\{\|h\|^2_{\mathrm{L}^2(0,\infty;\mathrm{L}^2)}\right\} \\ \lim_{\epsilon\to 0} \epsilon \log \mathbb{P}_{\epsilon}^{|1} = -\frac{1}{2} \inf_{h\in \mathrm{L}^2(0,\infty;\mathrm{L}^2): \int_{-l}^{l} |\mathbf{S}(h)(T,x)|^2 dx < 4(1-\gamma)} \left\{\|h\|^2_{\mathrm{L}^2(0,\infty;\mathrm{L}^2)}\right\} \end{array}$$

where $\mathbf{S}(h)$ and $\tilde{\mathbf{S}}(h)$ correspond to the usual skeleton with respectively a soliton and a null initial datum. Moreover, both infima are positive numbers.

Proof. The mapping φ from $\mathcal{X}_\infty$ into $\mathbb{R}^+$ such that $\varphi(f) = \int_{-l}^{l} |f(T,x)|^2 dx$ is continuous. Therefore, the direct image measures $\left(\varphi_*\mu^{u^\epsilon}\right)_{\epsilon\geq 0}$ and $\left(\varphi_*\mu^{u^{\epsilon,n}}\right)_{\epsilon\geq 0}$ satisfy LDP of speed ϵ and good rate functions respectively

$$I^T(y) = \frac{1}{2} \inf_{h\in \mathrm{L}^2(0,\infty;\mathrm{L}^2): \int_{-l}^{l} |\mathbf{S}(h)(T,x)|^2 dx = y} \left\{\|h\|^2_{\mathrm{L}^2(0,\infty;\mathrm{L}^2)}\right\}$$

and J^T where $\mathbf{S}$ is replaced by $\tilde{\mathbf{S}}$. Consequently,

$$\forall i \in \{0,1\},\ -L^i(\gamma) \leq \underline{\lim}_{\epsilon\to 0} \epsilon \log \mathbb{P}_{\epsilon}^{|i} \leq \overline{\lim}_{\epsilon\to 0} \epsilon \log \mathbb{P}_{\epsilon}^{|i} \leq -U^i(\gamma)$$

where

$$\begin{array}{ll} L^0(\gamma) = \inf_{y\in(4(1-\gamma),\infty)} J^T(y), & U^0(\gamma) = \inf_{y\in[4(1-\gamma),\infty)} J^T(y), \\ L^1(\gamma) = \inf_{y\in[0,4(1-\gamma))} I^T(y), & U^1(\gamma) = \inf_{y\in[0,4(1-\gamma)]} I^T(y). \end{array}$$

For every $\delta > 0$, $U^0(\gamma) \leq L^0(\gamma) \leq U^0(\gamma-\delta)$ and $U^1(\gamma) \leq L^1(\gamma) \leq U^1(\gamma+\delta)$ hold.

The function $\gamma \mapsto U^0(\gamma)$ is positive and decreasing. Also, since the range of Φ is dense, there exists a sequence $(h_n^0)_{n\in\mathbb{N}}$ of functions of $\mathrm{L}^2(0,\infty;\mathrm{L}^2)$ so that Φh_n converges to

$$H^0(t) = i\frac{\mathrm{d}u^0}{\mathrm{d}t} - \Delta u^0 - \lambda |u^0|^{2\sigma} u^0$$

where

$$u^0(t) = \mathbb{1}_{t\leq T}\frac{t}{T}\Psi_1(0,\cdot)$$

and by the continuity proved in Section A.5.1 $\left(\varphi \circ \tilde{\mathbf{S}}(h_n^0)\right)_{n\in\mathbb{N}}$ converges to $\varphi \circ \tilde{\mathbf{S}}(H^0) > 4\left(1 - \frac{\gamma_0}{2}\right) > 4(1-\gamma_0)$. Consequently, h_n^0 belongs to the minimizing set for n large enough. Thus, $U^0(\gamma_0) < \infty$ follows. Consequently, the function $\gamma \mapsto U^0(\gamma)$ possesses an at most countable set of points of discontinuity.

Similarly, the function $\gamma \mapsto U^1(\gamma)$ is a bounded increasing function. Also, if $(h_n^1)_{n\in\mathbb{N}}$ and $H^1(t)$ are defined as previously replacing $u^0(t)$ by

$$u^1(t) = \mathbb{1}_{t\leq T}\left(1 - \left(1 - \sqrt{\frac{\gamma_0}{2}}\right)\frac{t}{T}\right)\Psi_1(0,\cdot),$$

the sequence $(\varphi \circ \mathbf{S}(h_n^1))_{n\in\mathbb{N}}$ converges to $\varphi \circ \mathbf{S}(H^1) \leq 2\gamma_0 = 4\left(1 - \left(1 - \frac{\gamma_0}{2}\right)\right)$. Thus, for n large enough h_n^1 belongs to the minimizing set. Consequently, the function $\gamma \mapsto U^1(\gamma)$ has an at most countable set of points of discontinuity. Thus, for a well chosen γ, letting δ converge to zero, we obtain for $i \in \{0,1\}$ that $L^i(\gamma) = U^i(\gamma)$ and the equivalents follow.

From the arguments used in the proof of Proposition A.5.5, $\tilde{\mathbf{S}}$ is a continuous mapping from $\mathrm{L}^2(0,\infty;\mathrm{H}^1)$ into $\mathcal{X}_\infty$. Since φ is continuous, if $(H_n)_{n\in\mathbb{N}}$ is a sequence of functions converging to zero in $\mathrm{L}^2(0,\infty;\mathrm{H}^1)$ then $\left(\varphi \circ \tilde{\mathbf{S}}(H_n)\right)_{n\in\mathbb{N}}$ converges to $\varphi \circ \tilde{\mathbf{S}}(0) = 0$. Similarly $(\varphi \circ \mathbf{S}(H_n))_{n\in\mathbb{N}}$ converges to $\varphi \circ \mathbf{S}(0) > 4\left(1 - \frac{\gamma_0}{2}\right)$. We have now proved the last point of our result that is both infima are positive. □

In the two following sections we concentrate on the mass, we take $l = \infty$ as if the window were not bounded. Somehow, if we forget the coefficient, we concentrate on the tails of the marginal law of the soliton power when the initial datum is a soliton or the tails of the amplitude of the solution with null initial datum as ϵ converges to zero. We recall, it has been pointed out in the introduction, that the mass is no longer preserved in the stochastic case and is such that its expected value increases.

A.6.1 Upper bounds

The norm of the linear continuous operator Φ of L^2 is thereafter denoted by $\|\Phi\|_c$.

Proposition A.6.2 *For every positive T, γ in $(0,1)$, and every operator Φ in $\mathcal{L}_2(\mathrm{L}^2,\mathrm{H}^1)$, the inequalities*

$$\overline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{P}_\epsilon^{|0}\leq -\frac{1-\gamma}{2T\|\Phi\|_c^2}$$

and

$$\overline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{P}_\epsilon^{|1}\leq -\frac{\gamma^2}{2T\|\Phi\|_c^2(1+\gamma)}.$$

hold.

Proof. Multiplying by $-i\overline{u}$ the equation

$$i\frac{\mathrm{d}u}{\mathrm{d}t}-\Delta u-\lambda|u|^{2\sigma}u=\Phi h,$$

integrating over time and space and taking the real part gives that

$$\|u(T)\|_{\mathrm{L}^2}^2-\|u_0\|_{\mathrm{L}^2}^2=2\mathfrak{Re}\left(-i\int_0^T\int_{\mathbb{R}}\Phi h\overline{u}\;dxdt\right).$$

First bound: The boundary conditions $\|u(T)\|_{\mathrm{L}^2(\mathbb{R})}^2\geq 4(1-\gamma)$ and $u_0=0$ along with Cauchy Schwarz inequality imply both that

$$4(1-\gamma)\leq 2\|\Phi\|_c\|h\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}\|u\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)},$$

and that

$$\begin{aligned}\int_0^T\|u(t)\|_{\mathrm{L}^2}^2dt &= 2\int_0^T\mathfrak{Re}\left(-i\int_0^t\Phi h\overline{u}\;dxds\right)dt\\ &\leq 2T\|\Phi\|_c\|h\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}\|u\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)},\end{aligned}$$

thus,

$$\|h\|_{\mathrm{L}^2(0,\infty;\mathrm{L}^2)}^2\geq\frac{1-\gamma}{T\|\Phi\|_c^2}.$$

Second bound: The boundary conditions $\|u(T)\|_{\mathrm{L}^2}^2<4(1-\gamma)$ and $\|u_0\|_{\mathrm{L}^2}^2=4$ give both that along with the Cauchy-Schwarz inequality

$$4\gamma<2\|\Phi\|_c\|h\|_{\mathrm{L}^2(0,\infty;\mathrm{L}^2)}\|u\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}$$

and also along with Cauchy Schwarz and integration over time

$$\|u\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)} - 4T \leq 2T\|\Phi\|_c\|h\|_{\mathrm{L}^2(0,\infty;\mathrm{L}^2)}\|u\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}.$$

Consequently, it follows that

$$\|u\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \leq T\|\Phi\|_c\|h\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}\left(1+\sqrt{1+\frac{4}{T\|\Phi\|_c^2\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)}}}\right).$$

Thus, we obtain

$$\frac{2\gamma}{T\|\Phi\|_c^2} < \|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)}\left(1+\sqrt{1+\frac{4}{T\|\Phi\|_c^2\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)}}}\right)$$

and since the function $x \to x\left(1+\sqrt{1+\frac{4}{x}}\right)$ is increasing on $\mathbb{R}_+^*$,

$$\|h\|^2_{\mathrm{L}^2(0,\infty;\mathrm{L}^2)} > \frac{\gamma^2}{T\|\Phi\|_c^2(1+\gamma)}.$$

The upper bound follows. □

Remark A.6.3 *The estimates used in the proof of the above result only use the fact that the nonlinearity acts as a potential. Indeed the same result holds for any nonlinearity of this type.*

A.6.2 Lower bounds

We prove the following lower bounds.

Proposition A.6.4 *For every positive T, $\gamma \in (0,1)$ and $\tilde{\gamma}$ in to a dense subset of $(0,1)$, for every sequence of operators $(\Phi_n)_{n\in\mathbb{N}}$ in $\mathcal{L}_2^{0,1}$ such that for every h in $\mathrm{L}^2\left(0,T;\mathrm{L}^2\right)$ of the form*

$$h(t,x) = i\frac{\eta'(t)}{\eta(t)}\Psi_\eta(t,x) - i\sqrt{2}\eta'(t)\exp\left(-i\int_0^t \eta^2(s)ds\right)\eta(t)x\frac{\sinh}{\cosh^2}\left(\eta(t)x\right)$$

where

$$\Psi_\eta(t,x) = \sqrt{2}\eta(t)\exp\left(-i\int_0^t \eta^2(s)ds\right)\operatorname{sech}(\eta(t)x) \qquad \text{(A.6.1)}$$

and η is of one of the following parameterized form

$$\eta_{\tilde{\gamma},T}(t) = (1-\tilde{\gamma})\left(\frac{t}{T}\right)^2$$

or

$$\eta_{\tilde{\gamma},T}(t) = \left(2 - \tilde{\gamma} - 2\sqrt{1-\tilde{\gamma}}\right)\left(\frac{t}{T}\right)^2 + 2\left(-1 + \sqrt{1-\tilde{\gamma}}\right)\frac{t}{T} + 1,$$

$\Phi_n h$ *converges to* h *in* $\mathrm{L}^1(0,T;\mathrm{L}^2)$*, we obtain the following inequalities where the* n *in the error probabilities is there to recall that* Φ *is replaced by* Φ_n

$$\underline{\lim}_{n\to\infty,\epsilon\to 0}\epsilon \log \mathbb{P}^{|0}_{\epsilon,n} \geq -\frac{2(1-\gamma)(12+\pi^2)}{9T}$$

and

$$\underline{\lim}_{n\to\infty,\epsilon\to 0}\epsilon \log \mathbb{P}^{|1}_{\epsilon,n} \geq -\frac{2(1-\sqrt{1-\gamma})^2(12+\pi^2)}{9T}.$$

Proof. Consider first that "$\Phi = I$", denote the corresponding skeletons by $\tilde{\mathbf{S}}_{WN}$ when the initial datum is 0 and by $\mathbf{S}_{WN}$ when it is $\Psi(x) = \sqrt{2}\mathrm{sech}(x)$, they are defined from the (ii) of the Strichartz inequalities on $\mathrm{L}^1_{loc}\left(0,\infty;\mathrm{H}^1\right)$, suppose also that η is any function of $\mathrm{C}([0,T])$. Since the initial data 0 or Ψ belong to H^2, for $h \in \mathrm{L}^2$, $\mathbf{S}_{WN}(h)$ and $\tilde{\mathbf{S}}_{WN}(h)$ are functions of $\mathrm{C}([0,T];\mathrm{H}^2)\cap\mathrm{C}^1([0,T];\mathrm{L}^2)$, consequently $t \to \eta(t) = \frac{1}{4}\|\Psi_\eta(t,\cdot)\|^2_{\mathrm{L}^2}$ is necessarily a function in $\mathrm{C}^1([0,T])$. Also, for controls h_η parameterized as in the above assumptions, η is in $\mathrm{C}^1([0,T])$, the controls belong to $\mathrm{L}^1_{loc}\left(0,\infty;\mathrm{H}^1\right)$, the skeletons are the prescribed paths Ψ_η and we obtain

$$\begin{aligned}&\inf_{\eta\in\ \mathrm{C}^1([0,T]):\|\tilde{\mathbf{S}}_{WN}(h_\eta)(T,\cdot)\|^2_{\mathrm{L}^2}\geq 4(1-\gamma)}\left\{\|h_\eta\|^2_{\mathrm{L}^2(0,\infty;\mathrm{L}^2)}\right\}\\ &= \inf_{\eta\in\ \mathrm{C}^1([0,T]),b.c.}\int_0^T F\left(\eta(t),\eta'(t)\right)dt,\end{aligned}$$

where the Lagrangian F is

$$F(z,p) = \frac{1}{9}(12+\pi^2)\frac{p^2}{z},$$

and b.c. stands for the boundary conditions $\eta(0) = 0$ and $\eta(T) \geq 1-\gamma$. Indeed, since $\tilde{\mathbf{S}}_{WN}(h)(T)$ is a function of $(h(t))_{t\in[0,T]}$, the infimum could be taken on functions set to zero almost everywhere after T, thus $\|h\|^2_{\mathrm{L}^2(0,\infty;\mathrm{L}^2)}$ in the left hand side could be replaced by $\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)}$. A scaling argument gives that the terminal boundary condition is necessarily saturated.

Similarly, for the second error probability, $\tilde{\mathbf{S}}_{WN}$ is replaced by $\mathbf{S}_{WN}$ and b.c. is $\eta(0) = 1$ and $\eta(T) = 1-\gamma$.

The usual results of the indirect method do not apply to the problem of the calculus of variations, nonetheless solutions of the boundary value problem associated to the Euler-Lagrange equation

$$2\frac{\eta''}{\eta} = \left(\frac{\eta'}{\eta}\right)^2$$

provide upper bounds when we compute the integral of the Lagrangian. If we suppose that η is in $\mathrm{C}^3([0,T])$ and that it is positive on $(0,T)$, we obtain by derivation of the ODE that on $(0,T)$,

$$\eta''' = 0.$$

Also, looking for solutions of the form at^2+bt+c, we obtain that necessarily $b^2 = 4ac$. Thus $\mathrm{C}^3([0,T])$ positive solutions are necessarily of the form $a\left(t-\frac{b}{2a}\right)^2$. From the boundary conditions, we obtain that for the first error probability the function defined by

$$\eta^0(t) = (1-\gamma)\left(\frac{t}{T}\right)^2$$

is a solution of the boundary value problem. For the second error probability, the boundary conditions imply that the two following functions defined by

$$\eta^{1,1}(t) = \left(1+\sqrt{1-\gamma}\right)^2\left(\frac{t}{T}\right)^2 + 2\left(-1-\sqrt{1-\gamma}\right)\frac{t}{T}+1$$

and

$$\eta^{1,2}(t) = \left(1-\sqrt{1-\gamma}\right)^2\left(\frac{t}{T}\right)^2 + 2\left(-1+\sqrt{1-\gamma}\right)\frac{t}{T}+1$$

are solutions of the boundary value problem. The second function gives the smallest value when we compute the integral of the Lagrangian.

From the assumptions on the operators Φ_n and Lemma A.5.2, for functions h of the assumptions of the proposition, $(\mathcal{G}\circ\Lambda\circ\Phi_n)h$ converges to $(\mathcal{G}\circ\Lambda)h = \tilde{\mathbf{S}}_{WN}(h)$ in $\mathrm{C}\left([0,T];\mathrm{L}^2\right)$. In the above $\mathcal{G}$ is the mapping defined in Section A.2.5 with null initial datum. Thus $\|\tilde{\mathbf{S}}_n(h)(T,.)\|_{\mathrm{L}^2}^2$ converges to $\|\tilde{\mathbf{S}}_{WN}(h)(T,.)\|_{\mathrm{L}^2}^2$. As a consequence, for h in the particular parameterized set of controls where $\tilde{\gamma} = \gamma-\delta$ and $\delta > 0$, there exists N_0 large enough such that for any $n \geq N_0$, $\|\tilde{\mathbf{S}}_{WN}(h)(T,.)\|_{\mathrm{L}^2}^2 \geq 4(1-\gamma+\delta)$ implies that $\|\tilde{\mathbf{S}}_n(h)(T,.)\|_{\mathrm{L}^2}^2 > 4(1-\gamma)$. Thus the infimum in the rate for a fixed γ is smaller than the infimum on the smallest particular set of controls h, which is itself smaller than the square of the L^2-norm of the control h corresponding to $\eta_{\gamma,T}$. Indeed, h is such that $\|\tilde{\mathbf{S}}_{WN}(h)(T,.)\|_{\mathrm{L}^2}^2 \geq 4(1-\gamma+\delta)$, for n

large enough, which implies the expected boundary condition. We conclude from the upper bound obtained in the previous study of the problem of calculus of variations by guessing the likeliest path, by taking the limit inf in n of the opposite and from the fact that δ is then arbitrary. The end of the proof of the lower bound for the second error probability is the same. □

We have finally obtained upper and lower bounds that agree up to constants in their behavior in large T for the two error probabilities. In the case of the first error probability they also agree in their behavior in γ for γ near 1, it is not quite the case for the second error probability and γ near zero. The bounds for the first error probability are of the same order as the one we could obtain from the results of [62]. Indeed, with a slightly different normalization on the NLS equation and when the noise is the ideal white noise and thus $\|\Phi\|_c = 1$, a result of [62] is that the probability density function of the mass of the pulse at the coordinate T of the fiber when the initial datum is null is asymptotically that of an exponential law of parameter $\frac{\epsilon T}{2}$. Integrating the density over $[2(1-\gamma), \infty)$, we obtain that $\lim_{\epsilon\to 0} \epsilon \log \mathbb{P}_\epsilon^{|0} = -\frac{4(1-\gamma)}{T}$ which is indeed in between the two bounds and very close to the lower bound though obtained with a more general parametrization. Also in Section 6 of [63], the authors study numerically the second error probability by integrating the joint probability density function over a domain for the soliton power and timing jitter which depends on the size of the window and the threshold. They also consider the unrealistic case where the size of the window is large and of the order of the coordinate T of the location of the receiver in the fiber. The effect of the timing jitter could then be neglected and they obtain an error probability given by $\lim_{\epsilon\to 0} \epsilon \log \mathbb{P}_\epsilon^{|1} = -\frac{c(\gamma)}{T}$, with a constant $c(\gamma)$ which is a function of the threshold. It indeed exhibits the same behavior in T as the one obtained in our previous calculations.

Remark A.6.5 *In the proposition we would like to impose that the operator* Φ *is acting as the identity map on* $\operatorname{span}\left\{\frac{1}{\cosh(ax)}, x\frac{\sinh}{\cosh^2}(ax); a \in [0,1]\right\}$ *but it seems too strong to be compatible with the Hilbert-Schmidt assumption. On the other hand, we may check that the assumptions made here can easily be fulfilled. Also, under these assumptions, the noise is as close as possible to the space-time white noise considered in [62, 63] that we are not able to treat mathematically.*

Remark A.6.6 *Note that it is natural to obtain that the opposite of the error probabilities are decreasing functions of* T*. Indeed, the higher is* T*,* *the less energy is needed to form a signal which mass gets above a fixed threshold at the coordinate T. Replacing above by under, we obtain the same result in the case of a soliton as initial datum. Consequently, the higher is T the higher the error probabilities get. Also, in the case of the first error probability, both upper and lower bounds in Proposition A.6.2 and Proposition A.6.4 are increasing functions of γ. Similarly, in the case of the second error probability, the bounds are decreasing functions of γ. This could be interpreted as the higher is the threshold, the more energy is needed to form a signal which mass gets above the threshold at the coordinate T and conversely in the case of a soliton as initial datum.*

Remark A.6.7 *The results obtained numerically for a parametrization by the amplitude solely without modulating the phase and other shape parameters or of the phase $\eta^2(t)t$ instead of $\int_0^t \eta^2(s)ds$ in* (A.6.1)*, gave less interesting lower bounds that did not exhibit the desired properties in T.*

Remark A.6.8 *In [62] the following parametrization of the solution*

$$\sqrt{2}\eta(t)\exp\left(i\beta(t)x + i\alpha(t) + i\tau(t)\right)\left[\operatorname{sech}(\eta(t)(x - y(t))) + v(t,x)\right],$$

where $\tau'(t) = \eta^2(t)$, is studied. The authors give a physical justification of the fact that for large T the field v could be neglected. This could be compared with the results on asymptotic stability for the deterministic nonlinear Schrödinger equation.

Appendix B

Uniform large deviations for the nonlinear Schrödinger equation with multiplicative noise

Abstract: Uniform large deviations at the level of the paths for the stochastic nonlinear Schrödinger equation are presented. The noise is a real multiplicative Gaussian noise, white in time and colored in space. The trajectory space allows blow-up. It is endowed with a topology analogous to a projective limit topology. Asymptotics of the tails of the blow-up time are obtained as corollaries.

B.1 Introduction

The nonlinear Schrödinger (NLS) equation with a power law nonlinearity is a generic model in many areas of physics among which solid state physics and optics. It describes the propagation of slowly varying envelopes of a wave packet in media with both nonlinear and dispersive responses. In some cases, spatial and temporal fluctuations of the parameters of the medium have to be considered to account for example for thermal fluctuations or inhomogeneities in the medium; see for example [10, 11, 59]. Sometimes the only source of fluctuation that has significant effect on the dynamics enters linearly as a random potential. In fiber optics it accounts for Raman coupling to the thermal phonon; see [59] for details.

In the following, the noise is the time derivative in the sense of distributions of a Wiener process $(W(t))_{t\in[0,\infty)}$ defined on some real separable Hilbert space of real valued functions. As the unitary group $(U(t))_{t\in\mathbb{R}}$ generated by $-i\Delta$ on H^1, space of complex valued functions, is an isometry, there is no smoothing effect in the Sobolev spaces based on L^2. We are thus unable to treat the space-time white noise often considered in physics. The noise is thus white in time and colored in space. With the Itô notations we write

$$i\mathrm{d}u - (\Delta u + \lambda|u|^{2\sigma}u)\mathrm{d}t = u \circ \mathrm{d}W. \tag{B.1.1}$$

The symbol $\circ$ stands for the Stratonovich product. It corresponds, in terms of the Itô product, to

$$i\mathrm{d}u - \left(\Delta u + \lambda|u|^{2\sigma}u - \frac{i}{2}uF_\Phi\right)\mathrm{d}t = u\mathrm{d}W, \tag{B.1.2}$$

where $F_\Phi(x) = \sum_{j\in\mathbb{N}} (\Phi e_j(x))^2$ for x in $\mathbb{R}^d$. When $\lambda = 1$ the nonlinearity is called focusing, otherwise it is defocusing.

The well posedness of the Cauchy problem associated to the deterministic, see [25, 136], and stochastic, see [37], NLS equations depends on the size of σ. If $\sigma < \frac{2}{d}$, the nonlinearity is subcritical and the Cauchy problem is globally well posed in L^2 or H^1. If $\sigma = \frac{2}{d}$, critical nonlinearity, or $\frac{2}{d} < \sigma < \frac{2}{d-2}$ when $d \geq 3$ ($\sigma > \frac{2}{d}$ when d=1,2), supercritical nonlinearity, the Cauchy problem is locally well posed in H^1. In this latter case, if the nonlinearity is defocusing, solutions are global. In the focusing case and for the deterministic equation some initial data yield global solutions while other yield solutions which blow up in finite time. Results on the influence of a multiplicative noise on the blow-up appeared in a series of numerical and theoretical papers; see for example [13, 43] and [39].

In this article we prove a sample path large deviation principle (LDP) for equation (B.1.2). We give an application to the blow-up time. This type of study has been done for a noise of additive type in [81] where applications to error in fiber optic soliton transmission are also given. Applications to the study of the diffusion in position of a soliton pulse are derived in [44]. Note that our approach allows to prove rigorously results obtained by heuristic arguments in the physics literature.

A first type of proof for a LDP when the noise is multiplicative uses an extension of Varadhan's contraction principle; see [53], [48][Proposition 4.2.3] and [66]. It requires a sequence of approximations of the measurable Itô map by continuous maps uniformly converging on the sets of levels of the initial good rate function less or equal to a positive constant and that the resulting sequence of family of laws are exponentially good approximations of the laws of the solutions. A second type is based on the estimate of Proposition B.4.1. In [5], LDP for diffusions that may blow up in finite time are proved this way. This second type of proof is generally adopted in the case of SPDEs; see [22, 27, 29, 116]. It is that of the present paper. Note that in [22, 29] the approach to the stochastic calculus is based on martingale measures whereas in [27, 116] it is infinite dimensional. A third type of proof is based on the continuity theorem of T. Lyons for rough paths in the $p-$variation topology; see [104] in the case of diffusions.

Uniformity with respect to initial data in compact sets allows to study the first exit time and the most probable exit points from a neighborhood of an asymptotically stable equilibrium point. We have studied the case of weakly damped stochastic nonlinear Schrödinger equations when the noise is of additive or multiplicative type. It will appear elsewhere. Uniform LDPs also yield LDPs for the family of invariant measures of Markov transition semi-groups defined by SPDEs, when the noise goes to zero and when the measures weakly converge to a Dirac mass on 0. Results on invariant measures for some stochastic NLS equations are given in [45]. Uniform LDPs are proved in [5, 48] for diffusions and in [27, 29, 116, 131] for particular SPDEs.

B.2 Preliminaries and statement of the result

Throughout the paper the following notations will be used. The constant C is generic and may take different values, even within the same line.

The set of positive integers and positive real numbers are denoted by $\mathbb{N}^*$ and $\mathbb{R}_+^*$, while the set of real numbers different from 0 is denoted by $\mathbb{R}^*$.

For $p \in \mathbb{N}^*$, L^p is the Lebesgue space of complex valued functions. For k in $\mathbb{N}^*$, $\mathrm{W}^{k,p}$ is the Sobolev space of L^p functions with partial derivatives up to level k, in the sense of distributions, in L^p. For $p = 2$ and s in $\mathbb{R}_+^*$, H^s is the Sobolev space of tempered distributions v of Fourier transform $\hat{v}$ such that $(1+|\xi|^2)^{s/2}\hat{v}$ belongs to L^2. We denote the spaces by $\mathrm{L}^p_{\mathbb{R}}$, $\mathrm{W}^{k,p}_{\mathbb{R}}$ and $\mathrm{H}^s_{\mathbb{R}}$ when the functions are real-valued. The space L^2 is endowed with the inner product $(u,v)_{\mathrm{L}^2} = \mathfrak{Re}\int_{\mathbb{R}^d} u(x)\overline{v}(x)dx$. If I is an interval of $\mathbb{R}$, $(E, \|\cdot\|_E)$ a Banach space and r belongs to $[1,\infty]$, then $\mathrm{L}^r(I;E)$ is the space of strongly Lebesgue measurable functions f from I into E such that $t \to \|f(t)\|_E$ is in $\mathrm{L}^r(I)$.

The space of linear continuous operators from B into $\tilde{B}$, where B and $\tilde{B}$ are Banach spaces is $\mathcal{L}_c\left(B,\tilde{B}\right)$. When $B = H$ and $\tilde{B} = \tilde{H}$ are Hilbert spaces, such an operator is Hilbert-Schmidt when $\sum_{j\in\mathbb{N}}\|\Phi e_j^H\|_{\tilde{H}}^2<\infty$. The set of such operators is denoted by $\mathcal{L}_2(H,\tilde{H})$, or $\mathcal{L}_2^{s,r}$ when $H=\mathrm{H}^s_{\mathbb{R}}$ and $\tilde{H}=\mathrm{H}^r_{\mathbb{R}}$. The previous sum is the square of a norm that we denote by $\|\Phi\|_{\mathcal{L}_2(H,\tilde{H})}$.

Recall that when A and B are two Banach spaces, $A\cap B$, where the norm of an element is the maximum of the norms in A and in B, is a Banach space.

We recall that a pair $(r(p),p)$ of positive numbers is called an admissible pair if p satisfies $2\leq p<\frac{2d}{d-2}$ when $d>2$ ($2\leq p<\infty$ when $d=2$ and $2\leq p\leq\infty$ when $d=1$) and $r(p)$ is such that $\frac{2}{r(p)}=d\left(\frac{1}{2}-\frac{1}{p}\right)$. Given an admissible pair $(r(p),p)$ and S and T such that $0\leq S<T$, the space

$$X^{(S,T,p)}=\mathrm{C}\left([S,T];\mathrm{H}^1\right)\cap\mathrm{L}^{r(p)}\left(S,T;\mathrm{W}^{1,p}\right),$$

denoted by $X^{(T,p)}$ when $S=0$, is of particular interest for the NLS equation.

Now, let us present the space of exploding paths, see [81] for the main properties. We add a point Δ to the space H^1 and embed the space with the topology such that its open sets are the open sets of H^1 and the complement in $\mathrm{H}^1\cup\{\Delta\}$ of the closed bounded sets of H^1. The set $\mathrm{C}([0,\infty);\mathrm{H}^1\cup\{\Delta\})$ is then well defined. We denote the blow-up time of f in $\mathrm{C}([0,\infty);\mathrm{H}^1\cup\{\Delta\})$ by $\mathcal{T}(f)=\inf\{t\in[0,\infty):f(t)=\Delta\}$, with the convention that $\inf\emptyset=\infty$. We may now define the set

$$\mathcal{E}(\mathrm{H}^1)=\left\{f\in\mathrm{C}([0,\infty);\mathrm{H}^1\cup\{\Delta\}):f(t_0)=\Delta\Rightarrow\forall t\geq t_0,\ f(t)=\Delta\right\},$$

it is endowed with the topology defined by the neighborhood basis

$$V_{T,r}(\varphi_1)=\left\{\varphi\in\mathcal{E}(\mathrm{H}^1):\mathcal{T}(\varphi)>T,\ \|\varphi_1-\varphi\|_{\mathrm{C}([0,T];\mathrm{H}^1)}\leq r\right\},$$

of φ_1 in $\mathcal{E}(\mathrm{H}^1)$ given $T<\mathcal{T}(\varphi_1)$ and r positive.

We define $\mathcal{A}(d)$ and $\tilde{\mathcal{A}}(d)$ by the sets $[2,\infty)$ when $d=1$ or $d=2$ and respectively $\left[2,\frac{2(3d-1)}{3(d-1)}\right)$ and $\left[2,\frac{2d}{d-1}\right)$ when $d\geq 3$. The space $\mathcal{E}_\infty$ is now defined for any d in $\mathbb{N}^*$ by the set of functions f in $\mathcal{E}(\mathrm{H}^1)$ such that for every $p\in\mathcal{A}(d)$ and all $T\in[0,\mathcal{T}(f))$, f belongs to $\mathrm{L}^{r(p)}\left(0,T;\mathrm{W}^{1,p}\right)$. It is endowed with the topology defined for φ_1 in $\mathcal{E}_\infty$ by the neighborhood basis

$$W_{T,p,r}(\varphi_1)=\{\varphi\in\mathcal{E}_\infty:\mathcal{T}(\varphi)>T,\ \|\varphi_1-\varphi\|_{X^{(T,p)}}\leq r\}$$

where $T<\mathcal{T}(\varphi_1)$, p belongs to $\mathcal{A}(d)$ and r is positive. The space is a Hausdorff topological space and thus we may consider applying generalizations of Varadhan's contraction principle. If we denote again by $\mathcal{T}:\mathcal{E}_\infty\to[0,\infty]$

the blow-up time, the mapping is measurable and lower semicontinuous.

We denote by $x \wedge y$ and by $x \vee y$ respectively the minimum and maximum of x and y. Recall that a rate function I is a lower semicontinuous function and that it is good if for every c positive, $\{x : I(x) \leq c\}$ is a compact set.

Let us recall the well known properties on the linear group $(U(t))_{t\in\mathbb{R}}$ generated by the deterministic linear equation. First for every $p \geq 2$, $t \neq 0$ and u_0 in $\mathrm{L}^{p'}$,

$$\|U(t)u_0\|_{\mathrm{W}^{1,p}} \leq (4\pi|t|)^{-d\left(\frac{1}{2}-\frac{1}{p}\right)}\|u_0\|_{\mathrm{W}^{1,p'}}, \quad \frac{1}{p}+\frac{1}{p'}=1. \tag{B.2.1}$$

We also have the Strichartz inequalities, see [136],

(i) There exists C positive such that for u_0 in H^1, T positive and $(r(p), p)$ admissible pair,

$$\|U(\cdot)u_0\|_{X^{(T,p)}} \leq C\,\|u_0\|_{\mathrm{L}^2}\,,$$

(ii) For every T positive, $(r(p), p)$ and $(r(q), q)$ admissible pairs, s and ρ such that $\frac{1}{s}+\frac{1}{r(q)}=1$ and $\frac{1}{\rho}+\frac{1}{q}=1$, there exists C positive such that for f in $\mathrm{L}^s\left(0,T;\mathrm{W}^{1,\rho}\right)$,

$$\left\|\int_0^\cdot U(\cdot-s)f(s)ds\right\|_{X^{(T,p)}} \leq C\|f\|_{\mathrm{L}^s(0,T;\mathrm{W}^{1,\rho})}.$$

We consider W originated from a cylindrical Wiener process on L^2, *i.e.* $W=\Phi W_c$ where Φ is Hilbert-Schmidt. It is known that for any orthonormal basis $(e_j)_{j\in\mathbb{N}}$ of $\mathrm{L}^2_{\mathbb{R}}$ there exists a sequence of real independent Brownian motions $(\beta_j)_{j\in\mathbb{N}}$ such that $W_c(t,x,\omega)=\sum_{j\in\mathbb{N}}\beta_j(t,\omega)e_j(x)$. In [37], the assumption on Φ is that it is $\gamma-$radonifying from $\mathrm{L}^2_{\mathbb{R}}$ into $\mathrm{W}^{1,\alpha}\cap\mathcal{L}_2^{0,1}$ where $\alpha > 2d$, this ensures that the noise lives in the Banach space $\mathrm{W}^{1,\alpha}\cap\mathrm{H}^1_{\mathbb{R}}$. In the proof of the continuity with respect to the potential on the sets of levels less or equal to a positive constant of the rate function of the sample path LDP for the Wiener process we use the continuous embedding of $\mathrm{H}^s_{\mathbb{R}}$ in $\mathrm{W}^{1,\infty}_{\mathbb{R}}$. In that respect, it would be enough to impose that Φ is $\gamma-$radonifying from L^2 into $\mathrm{H}^1_{\mathbb{R}}\cap\mathrm{W}^{1+s,\beta}$ for any $s\beta > d$. This embedding is again invoked in the proof of the exponential tail estimates, where in addition we use an expansion on a complete orthonormal system and thus that the image is a Hilbert space. Thus, we make the stronger assumption (A) which is given in terms of Hilbert spaces

$$\text{for some } s > \tfrac{d}{2}+1,\ \Phi\in\mathcal{L}_2^{0,s}. \tag{A}$$

We finally assume that the probability space is endowed with the filtration $\mathcal{F}_t = \mathcal{N} \cup \sigma\{W_s, 0 \leq s \leq t\}$ where $\mathcal{N}$ denotes the $\mathbb{P}-$null sets.

In the following we restrict ourselves to the case where $\frac{1}{2} \leq \sigma$ if $d = 1, 2$ or $\frac{1}{2} \leq \sigma < \frac{2}{d-2}$ if $d \geq 3$. We are interested, for ϵ positive, in the mild solutions

$$i\mathrm{d}u^{\epsilon,u_0} = \left(\Delta u^{\epsilon,u_0} + \lambda |u^{\epsilon,u_0}|^{2\sigma} u^{\epsilon,u_0} - \frac{i\epsilon}{2} F_\Phi u^{\epsilon,u_0}\right) \mathrm{d}t + \sqrt{\epsilon} u^{\epsilon,u_0} \mathrm{d}W,$$

with initial datum $u_0 \in \mathrm{H}^1$, *i.e.* solutions of the integral equation

$$\begin{array}{rl} u^{\epsilon,u_0}(t) = & U(t)u_0 - \int_0^t U(t-s) \left(i\lambda |u^{\epsilon,u_0}(s)|^{2\sigma} u^{\epsilon,u_0}(s) + \frac{\epsilon}{2} u^{\epsilon,u_0}(s) F_\Phi\right) ds \\ & -i\sqrt{\epsilon} \int_0^t U(t-s) u^{\epsilon,u_0}(s) dW(s). \end{array}$$

Recall that $F_\Phi(x) = \sum_{j\in\mathbb{N}} (\Phi e_j(x))^2$. Note that from the assumptions on Φ, F_Φ belongs to $\mathrm{W}^{1,\infty}_{\mathbb{R}}$. The product with u^{ϵ,u_0} in H^1 is thus well defined. For the weaker assumptions on Φ in [37], the "convolution" of the product is meaningful in the space considered for the fixed point, proving the local well posedness, thanks to the (ii) of the Strichartz inequalities.

Since the stronger the topology, the sharper the estimates, we take advantage of the variety of spaces where the fixed point can be conducted, as it has been done in [81], due to the integrability property and state the LDP in $\mathcal{E}_\infty$. Note that we may check quite easily from the fixed point argument in [37] that the solutions belong to $\mathcal{E}_\infty$. Also, using similar arguments as in [81], it can be shown that the solutions define random variables with values in $\mathcal{E}_\infty$. The laws are denoted by $\mu^{u^{\epsilon,u_0}}$. The rate function of the LDP for these family of measures is the infimum of the L^2-norm of the controls, of the control problem

$$\left\{ \begin{array}{l} i\frac{\mathrm{d}u}{\mathrm{d}t} = \Delta u + \lambda |u|^{2\sigma} u + u\Phi h, \;\; h \in \mathrm{L}^2\left(0, \infty; \mathrm{L}^2\right), \\ u(0) = u_0 \;\; \in \mathrm{H}^1, \end{array} \right.$$

producing the prescribed path. We may write the mild solution, or skeleton,

$$\mathbf{S}^c(u_0, h) = U(t)u_0 - i\int_0^t U(t-s) \left[\mathbf{S}^c(u_0, h)(s) \left(\lambda |\mathbf{S}^c(u_0, h)(s)|^{2\sigma} + \Phi h(s)\right)\right] ds.$$

If we replace Φh by $\frac{\partial f}{\partial t}$ where f belongs to $\mathrm{H}^1_0(0, \infty; \mathrm{H}^s_{\mathbb{R}})$ which is the subspace of $\mathrm{C}([0, \infty); \mathrm{H}^s_{\mathbb{R}})$ of locally square integrable in time and with locally square integrable in time time derivative functions, with null initial datum, we write

$$\mathbf{S}(u_0, f) = U(t)u_0 - i\int_0^t U(t-s) \left[\mathbf{S}(u_0, f)(s) \left(\lambda |\mathbf{S}(u_0, f)(s)|^{2\sigma} + \frac{\partial f}{\partial s}\right)\right] ds.$$

The topology on $\mathrm{C}\left([0,\infty);\mathrm{H}^s_{\mathbb{R}}\right)$ is that of the uniform convergence on the compact subsets of $[0,\infty)$. In the following we write $K \subset\subset \mathrm{H}^1$ when K is a compact subset of H^1 and by $Int(A)$ the interior of A.

Theorem B.2.1 *The family $\left(\mu^{u^{\epsilon,u_0}}\right)_{\epsilon>0}$ satisfies a uniform LDP of speed ϵ and good rate function*

$$\begin{aligned} I^{u_0}(w) &= \inf_{f\in \mathrm{H}^1_0\left(0,\infty;\mathrm{H}^s_{\mathbb{R}}\right):w=\mathbf{S}(u_0,f)} I^W(f) \\ &= \tfrac{1}{2}\inf_{h\in \mathrm{L}^2(0,\infty;\mathrm{L}^2):w=\mathbf{S}^c(u_0,h)} \|h\|^2_{\mathrm{L}^2(0,\infty;\mathrm{L}^2)}, \end{aligned}$$

where I^W is the rate function of the sample path LDP for the Wiener process, i.e. $\forall K \subset\subset \mathrm{H}^1,\ \forall A \in \mathcal{B}\left(\mathcal{E}_\infty\right)$,

$$\begin{aligned} &-\sup_{u_0\in K}\inf_{w\in Int(A)} I^{u_0}(w) \le \underline{\lim}_{\epsilon\to 0}\epsilon \log \inf_{u_0\in K}\mathbb{P}\left(u^{\epsilon,u_0}\in A\right) \\ &\le \overline{\lim}_{\epsilon\to 0}\epsilon\log\sup_{u_0\in K}\mathbb{P}\left(u^{\epsilon,u_0}\in A\right) \le -\inf_{w\in\overline{A},u_0\in K} I^{u_0}(w). \end{aligned}$$

This result is proved in Section B.3 and B.4. The aim is to push forward the following sample path LDP for the Wiener process. It follows from the general LDP for Gaussian measures on a real separable Banach space, see [53], the fact that the laws of the restrictions of $\sqrt{\epsilon}W$ on $\mathrm{C}\left([0,T];\mathrm{H}^s_{\mathbb{R}}\right)$ and $\mathrm{L}^2\left(0,T;\mathrm{H}^s_{\mathbb{R}}\right)$ have same RKHS and Dawson-Gärtner's theorem for projective limits.

Proposition B.2.2 *The family of laws of $(\sqrt{\epsilon}W)_{\epsilon>0}$ on $\mathrm{C}([0,\infty);\mathrm{H}^s_{\mathbb{R}})$ satisfies a LDP of speed ϵ and good rate function*

$$I^W(f) = \frac{1}{2}\inf_{h\in\mathrm{L}^2(0,\infty;\mathrm{L}^2):f=\int_0^\cdot \Phi h(s)ds}\|h\|^2_{\mathrm{L}^2(0,\infty;\mathrm{L}^2)}.$$

We denote by C_a the set $\left\{f\in\mathrm{C}\left([0,\infty);\mathrm{H}^s_{\mathbb{R}}\right) : I^W(f)\le a\right\}$ also equal to

$$\left\{f\in\mathrm{H}^1_0\left(0,\infty;\mathrm{H}^s_{\mathbb{R}}\right) : f(0)=0,\ \frac{\partial f}{\partial t}\in \mathrm{Im}\Phi,\ \left\|\Phi^{-1}_{|(\mathrm{Ker}\Phi)^\perp}\frac{\partial f}{\partial t}\right\|_{\mathrm{L}^2(0,\infty;\mathrm{L}^2)}\le\sqrt{2a}\right\}.$$

B.3 Continuity of the skeleton with respect to the control on the sets C_a and exponential tail estimates

We start by a result on the continuity of the skeleton with respect to the control on the sets C_a. It is used to prove the lower bound of the LDP and that the rate function is good; see Section B.5. In that respect, our proof is closer to the proofs in [22, 29]. The authors of [27, 116] use some slightly different arguments and prove separately the compactness of the set of levels of the rate function less or equal to a positive constant, the lower and upper bounds of the LDP using their characterizations in metric spaces. Note that in the applications, B.6, we need to compute infima of a quantity involving the rate function and the continuity proves to be very useful. We prove the continuity with respect to the control and initial datum as suggested in [29] though the continuity with respect to the initial data is not used in the proof of the uniform LDP.

Proposition B.3.1 *For every $u_0 \in \mathrm{H}^1$, a positive and f in C_a, $\mathbf{S}(u_0, f)$ exists and is uniquely defined. It is a continuous mapping from $\mathrm{H}^1 \times C_a$ into $\mathcal{E}_\infty$, where C_a has the topology induced by that of $\mathrm{C}\left([0,\infty); \mathrm{H}^s_{\mathbb{R}}\right)$.*

Proof. Let $\mathfrak{F}$ denote the mapping such that

$$\mathfrak{F}(u, u_0, f) = U(t)u_0 - i \int_0^t U(t-s) \left[u(s) \left(\lambda |u(s)|^{2\sigma} + \frac{\partial f}{\partial s} \right) \right] ds.$$

Let a and r be positive, f in C_a and u_0 such that $\|u_0\|_{\mathrm{H}^1} \le r$, set $R = 2cr$ where c is the norm of the continuous mapping of the (i) of the Strichartz inequalities. From (i) and (ii) of the Strichartz inequalities along with Hölder's inequality, the Sobolev injections and the continuity of Φ,
for any T positive, p in $\mathcal{A}(d)$, u and v in $X^{(T,p)}$ and any ν in $\left(0, 1 - \frac{\sigma(d-2)}{2}\right)$, it is a well defined interval since $\sigma < \frac{2}{d-2}$, there exists C positive such that

$$\begin{aligned}
&\|\mathfrak{F}(u, u_0, f)\|_{X^{(T,p)}} \\
&\le c\|u_0\|_{\mathrm{H}^1} + CT^\nu \|u\|^{2\sigma+1}_{X^{(T,p)}} + CT^{\frac{1}{2} - \frac{1}{r(p)}} \|u\|_{\mathrm{C}([0,T];\mathrm{H}^1)} \left\| \frac{\partial f}{\partial s} \right\|_{\mathrm{L}^2\left(0,T;\mathrm{W}^{1,\frac{r(p)d}{2}}\right)} \\
&\le c\|u_0\|_{\mathrm{H}^1} + CT^\nu \|u\|^{2\sigma+1}_{X^{(T,p)}} + C\sqrt{a} T^{\frac{1}{2} - \frac{1}{r(p)}} \|u\|_{X^{(T,p)}}.
\end{aligned}$$

Similarly we obtain

$$\begin{aligned}
&\|\mathfrak{F}(u, u_0, f) - \mathfrak{F}(v, u_0, f)\|_{X^{(T,p)}} \\
&\le C \left[T^\nu \left(\|u\|^{2\sigma}_{X^{(T,p)}} + \|v\|^{2\sigma}_{X^{(T,p)}} \right) + T^{\frac{1}{2} - \frac{1}{r(p)}} \sqrt{a} \right] \|u - v\|_{X^{(T,p)}}.
\end{aligned}$$

Thus, for $T = T^*_{r,a,p}$ small enough depending on r, a and p, the ball centered at 0 of radius R is invariant and the mapping $\mathfrak{F}(\cdot, u_0, f)$ is a $\frac{3}{4}-$contraction. We denote by $\mathbf{S}^0(u_0, f)$ the unique fixed point of $\mathfrak{F}(\cdot, u_0, f)$ in $X^{(T^*_{r,a,p},p)}$.

Also when T is positive, we can solve the fixed point problem on any interval $\left[kT^*_{r,a,p},(k+1)T^*_{r,a,p}\right]$ with $1 \le k \le \left\lfloor \frac{T}{T^*_{r,a,p}} \right\rfloor$. The fixed point is denoted by $\mathbf{S}^k(u_k,f)$ where $u_k=\mathbf{S}^{k-1}(u_{k-1},f)(kT^*_{r,a,p})$, as long as $\|u_k\|_{\mathrm{H}^1} \le r$. Existence and uniqueness of a maximal solution $\mathbf{S}(u_0,f)$ follows. It coincides with $\mathbf{S}^k(u_k,f)$ on the above intervals when it is defined. We may also show that the blow-up time corresponds to the blow-up of the H^1-norm, thus $\mathbf{S}(u_0,f)$ is an element of $\mathcal{E}_\infty$.

We shall now prove the continuity. Take u_0 in H^1, a positive and f in C_a. From the definition of the neighborhood basis of the topology of $\mathcal{E}_\infty$, it is enough to see that for ϵ positive, $T < \mathcal{T}\left(\mathbf{S}(u_0,f)\right)$, and $p \in \mathcal{A}(d)$, there exists η positive such that for every $\tilde{u}_0$ in H^1 and g in C_a satisfying $\|u_0-\tilde{u}_0\|_{\mathrm{H}^1}+\|f-g\|_{\mathrm{C}\left([0,T];\mathrm{H}^s_{\mathbb{R}}\right)} \le \eta$ then $\|\mathbf{S}(u_0,f)-\mathbf{S}(\tilde{u}_0,g)\|_{X^{(T,p)}} \le \epsilon$. We set $r=\|\mathbf{S}(u_0,f)\|_{X^{(T,p)}}+1,\ N=\left\lfloor \frac{T}{T^*_{r,a,p}} \right\rfloor,\ \delta_{N+1}=\frac{\epsilon}{N+1}\wedge 1$, and define for $k\in\{0,...,N\}$, δ_k and η_k such that $0<\delta_k<\delta_{k+1}$, $0<\eta_k<\eta_{k+1}<1$ and

$$\left\|\mathbf{S}^{k+1}(u_k,f)-\mathbf{S}^{k+1}(\tilde{u}_k,g)\right\|_{X^{(kT^*_{r,a,p},(k+1)T^*_{r,a,p},p)}} \le \delta_{k+1},$$

if

$$\|u_k-\tilde{u}_k\|_{\mathrm{H}^1}+d_{\mathrm{C}\left([0,\infty);\mathrm{H}^s_{\mathbb{R}}\right)}(f,g) \le \eta_{k+1}.$$

The distance $d_{\mathrm{C}\left([0,\infty);\mathrm{H}^s_{\mathbb{R}}\right)}(\cdot,\cdot)$ is one of the classical distances of the topology of uniform convergence on compact subsets of $[0,\infty)$. It is possible to choose $(\delta_k)_{k=0,...,N}$ as long as we prove the continuity for $k=0$. The maximal solution $\mathbf{S}(\tilde{u}_0,g)$ then necessarily satisfies $\mathcal{T}\left(\mathbf{S}(\tilde{u}_0,g)\right)>T$. We conclude setting $\eta=\eta_1$ and using the triangle inequality.

We now prove the continuity for $k=0$. First note that $\|u_0\|_{\mathrm{H}^1}\le r$ and $\|\tilde{u}_0\|_{\mathrm{H}^1}\le r$ since $\eta<1$, thus if Υ_1 denotes $\left\|\mathbf{S}^0(u_0,f)-\mathbf{S}^0(\tilde{u}_0,g)\right\|_{X^{(T^*_{r,a,p},p)}}$,

$$\begin{aligned}\Upsilon_1 &= \left\|\mathfrak{F}(\mathbf{S}^0(u_0,f),u_0,f)-\mathfrak{F}(\mathbf{S}^0(\tilde{u}_0,g),\tilde{u}_0,g)\right\|_{X^{(T^*_{r,a,p},p)}}\\ &\le \left\|\mathfrak{F}(\mathbf{S}^0(u_0,f),u_0,f)-\mathfrak{F}(\mathbf{S}^0(u_0,f),\tilde{u}_0,g)\right\|_{X^{(T^*_{r,a,p},p)}}\\ &\quad+\left\|\mathfrak{F}(\mathbf{S}^0(u_0,f),\tilde{u}_0,g)-\mathfrak{F}(\mathbf{S}^0(\tilde{u}_0,g),\tilde{u}_0,g)\right\|_{X^{(T^*_{r,a,p},p)}},\end{aligned}$$

and since $\mathfrak{F}(\cdot,\tilde{u}_0,g)$ is a $\frac{3}{4}-$contraction,

$$\begin{aligned}\Upsilon_1 &\le 4\left\|\mathfrak{F}(\mathbf{S}^0(u_0,f),u_0,f)-\mathfrak{F}(\mathbf{S}^0(u_0,f),\tilde{u}_0,g)\right\|_{X^{(T^*_{r,a,p},p)}}\\ &\le 4\left(c\|u_0-\tilde{u}_0\|_{\mathrm{H}^1}+\Upsilon_2\right)\end{aligned}$$

where, using Hölder's inequality and taking $p<\tilde{p}$,

$$\Upsilon_2 = \left\|\int_0^{\cdot} U(\cdot-s)\mathbf{S}^0(u_0,f)\frac{\partial(f-g)}{\partial s}(s)ds\right\|_{X^{(T^*_{r,a,p},p)}} \le \Upsilon_3\vee\Upsilon_4$$

and

$$\begin{aligned}\Upsilon_3 &= \left(\left\| \int_0^{\cdot} U(\cdot - s)\mathbf{S}^0(u_0,f)\frac{\partial(f-g)}{\partial s}(s)ds \right\|^{\theta}_{\mathrm{C}\left(\left[0,T^*_{r,a,p}\right];\mathrm{H}^1\right)} \right.\\ &\quad \left. \times \left\| \int_0^{\cdot} U(\cdot - s)\mathbf{S}^0(u_0,f)\frac{\partial(f-g)}{\partial s}(s)ds \right\|^{1-\theta}_{\mathrm{L}^{r(\tilde{p})}\left(0,T^*_{r,a,p};\mathrm{W}^{1,\tilde{p}}\right)} \right),\end{aligned}$$

with $\theta = \frac{\tilde{p}-p}{\tilde{p}-2}$,

$$\Upsilon_4 = \left\| \int_0^{\cdot} U(\cdot - s)\mathbf{S}^0(u_0,f)\frac{\partial(f-g)}{\partial s}(s)ds \right\|_{\mathrm{C}\left(\left[0,T^*_{r,a,p}\right];\mathrm{H}^1\right)}.$$

From the (ii) of the Strichartz inequalities we obtain

$$\Upsilon_4 \leq C\left(\sqrt{a}\left(T^*_{r,a,p}\right)^{\frac{1}{2}-\frac{1}{r(\tilde{p})}} R\right)^{1-\theta}.$$

It is now enough to show that when g is close enough to f, Υ_3 can be made arbitrarily small. Take n in $\mathbb{N}$ and set for i in $\{0,...,n\}$, $t_i = \frac{iT^*_{r,a,p}}{n}$ and

$$\mathbf{S}^{0,n}(u_0,f) = U(t-t_i)\left(\mathbf{S}(u_0,f)(t_i)\right), \text{ for } t_i \leq t < t_{i+1}. \tag{B.3.1}$$

As in the previous calculations we obtain

$$\begin{aligned}&\left\| \int_0^{\cdot} U(\cdot - s)\left(\mathbf{S}^0(u_0,f) - \mathbf{S}^{0,n}(u_0,f)\right)\frac{\partial(f-g)}{\partial s}(s)ds \right\|_{\mathrm{C}\left(\left[0,T^*_{r,a,p}\right];\mathrm{H}^1\right)}\\ &\leq C\sqrt{a}\left(T^*_{r,a,p}\right)^{\frac{1}{2}-\frac{1}{r(p)}} \left\|\mathbf{S}^0(u_0,f) - \mathbf{S}^{0,n}(u_0,f)\right\|_{\mathrm{C}\left(\left[0,T^*_{r,a,p}\right];\mathrm{H}^1\right)}\\ &\leq C\sqrt{a}\left(T^*_{r,a,p}\right)^{\frac{1}{2}-\frac{1}{r(p)}}\\ &\sup_{i\in\{0,...,n-1\}} \left\| \int_{t_i}^{\cdot} U(\cdot - s)\left[\mathbf{S}^0(u_0,f)\left(\lambda\left|\mathbf{S}^0(u_0,f)\right|^{2\sigma} + \frac{\partial f}{\partial s}\right)\right](s)ds \right\|_{\mathrm{C}([t_i,t_{i+1}];\mathrm{H}^1)}\\ &\leq C\sqrt{a}\left(T^*_{r,a,p}\right)^{\frac{1}{2}-\frac{1}{r(p)}}\left[R^{2\sigma+1}\left(\frac{T^*_{r,a,p}}{n}\right)^{\nu} + \left(\frac{T^*_{r,a,p}}{n}\right)^{\frac{1}{2}-\frac{1}{r(p)}} R\sqrt{a}\right].\end{aligned}$$

It can be made arbitrarily small for large n. Now it remains to bound the $\mathrm{C}\left(\left[0,T^*_{r,a,p}\right];\mathrm{H}^1\right)$ −norm of

$$\begin{aligned}&\int_0^t U(t-s)\mathbf{S}^{0,n}(u_0,f)\frac{\partial(f-g)}{\partial s}(s)ds\\ &= \sum_{i=0}^{n-1}\int_{t_i\wedge t}^{t_{i+1}\wedge t} U(t-t_i\wedge t)\left(\mathbf{S}^0(u_0,f)(t_i\wedge t)\frac{\partial(f-g)}{\partial s}(s)\right)ds\\ &= \sum_{i=0}^{n-1} U(t-t_i\wedge t)\left[\mathbf{S}^0(u_0,f)(t_i\wedge t)\left((f-g)(t_{i+1}\wedge t) - (f-g)(t_i\wedge t)\right)\right]\end{aligned}$$

since, from the Sobolev injections and the fact that $U(t-t_i\wedge t)$ is a group on H^1, for any i in $\{0,...,n-1\}$ and v in $\mathrm{H}^s_{\mathbb{R}}$,

$$\left\|U(t-t_i\wedge t)\left(\mathbf{S}^0(u_0,f)(t_i\wedge t)v\right)\right\|_{\mathrm{H}^1} \leq C\left\|\mathbf{S}^0(u_0,f)(t_i\wedge t)\right\|_{\mathrm{H}^1}\|v\|_{\mathrm{H}^s_{\mathbb{R}}}$$

and as $\frac{\partial(f-g)}{\partial s}$ is Bochner integrable, we obtain an upper bound of the form $rCn\|f-g\|_{C\left([0,T];H^s_{\mathbb{R}}\right)}$. For fixed n, it can be made arbitrarily small taking g sufficiently close to f. $\square$

We now give the exponential tail estimates.

Lemma B.3.2 *Assume that ξ is a point-predictable process, that p belongs to $\tilde{A}(d)$, T is positive and that there exists η positive such that $\|\xi\|^2_{C([0,T];H^1)} \leq \eta$ a.s., then for every t in $[0,T]$ and δ positive,*

$$\mathbb{P}\left(\sup_{t_0\in[0,T]}\left\|\int_0^{t_0} U(t-s)\xi(s)dW(s)\right\|_{W^{1,p}} \geq \delta\right) \leq \exp\left(1-\frac{\delta^2}{\kappa(\eta)}\right),$$

where

$$\kappa(\eta) = \frac{4c\left(\frac{r(p)d}{2}\right)^2 T^{1-\frac{4}{r(p)}}(d+1)(d+p)\|\Phi\|^2_{\mathcal{L}_2^{0,s}}}{1-\frac{4}{r(p)}}\eta,$$

and $c\left(\frac{r(p)d}{2}\right)$ is the norm of the continuous embedding $H^s_{\mathbb{R}} \subset W^{1,\frac{r(p)d}{2}}_{\mathbb{R}}$.

Proof. Let us denote by $g_a(f) = \left(1+a\|f\|^p_{W^{1,p}}\right)^{\frac{1}{p}}$ the real-valued function parameterized by a positive and by M the martingale defined by $M(t_0) = \int_0^{t_0} U(t-s)\xi(s)dW(s)$. The function g_a is twice Fréchet differentiable, the first and second derivatives at point $M(t)$ in the direction h are denoted by $Dg_a(M(t)).h$ and $D^2g_a(M(t),M(t)).(h,h)$, they are continuous. Also, the second derivative is uniformly continuous on the bounded sets. By the Itô formula the following decomposition holds

$$g_a(M(t_0)) = 1 + E_a(t_0) + \frac{1}{2}R_a(t_0)$$

where $E_a(t_0)$ is equal to

$$\int_0^{t_0} Dg_a(M(s)).U(t-s)\xi(s)dW(s) - \frac{1}{2}\int_0^{t} \|Dg_a(M(s)).U(t-s)\xi(s)\|^2_{\mathcal{L}_2(L^2,\mathbb{R})}ds$$

and $R_a(t_0)$ to

$$\begin{aligned}&\textstyle\int_0^{t_0} \|Dg_a(M(s)).U(t-s)\xi(s)\|^2_{\mathcal{L}_2(L^2,\mathbb{R})}ds\\ &+\textstyle\int_0^{t_0}\sum_{j\in\mathbb{N}} D^2g_a(M(s),M(s)).(U(t-s)\xi(s)\Phi e_j, U(t-s)\xi(s)\Phi e_j)ds,\end{aligned}$$

where $(e_j)_{j\in\mathbb{N}}$ is a complete orthonormal system of L^2. We denote by

$$q(u,h) \;\; = \mathfrak{Re}\left[\textstyle\int_{\mathbb{R}^d} \overline{u}|u|^{p-2}hdx + \sum_{k=1}^d \int_{\mathbb{R}^d} \overline{\partial_{x_k}u}|\partial_{x_k}u|^{p-2}\partial_{x_k}hdx\right],$$

then $Dg_a(u).h = a\left(1+a\|u\|_{\mathrm{W}^{1,p}}^p\right)^{\frac{1}{p}-1} q(u,h)$, and

$$\begin{array}{l} D^2 g_a(u,u).(h,g) = a^2(1-p)\left(1+a\|u\|_{\mathrm{W}^{1,p}}^p\right)^{\frac{1}{p}-2} q(u,h)q(u,g) \\ +a\left(1+a\|u\|_{\mathrm{W}^{1,p}}^p\right)^{\frac{1}{p}-1}\left[\left(1+\frac{p-2}{2}\right)\mathfrak{Re}\int_{\mathbb{R}^d}\left(|u|^{p-2}\overline{g}h+\sum_{k=1}^d |\partial_{x_k}u|^{p-2}\overline{\partial_{x_k}g}\partial_{x_k}h\right)dx \right. \\ \left. +\frac{p-2}{2}\mathfrak{Re}\int_{\mathbb{R}^d}\left(\overline{u}^2|u|^{p-4}gh+\sum_{k=1}^d \overline{\partial_{x_k}u}^2|\partial_{x_k}u|^{p-4}\partial_{x_k}g\partial_{x_k}h\right)dx\right]. \end{array}$$

From the series expansion of the Hilbert-Schmidt norm along with Hölder's inequality, we obtain that $R_a(t_0)$ is less than

$$\begin{array}{l} a(d+1)\int_0^{t_0}\left[a(d+1)\sum_{j\in\mathbb{N}}\|U(t-s)\xi(s)\Phi e_j\|_{\mathrm{W}^{1,p}}^2\left(\frac{\|M(s)\|_{\mathrm{W}^{1,p}}}{\left(1+a\|M(s)\|_{\mathrm{W}^{1,p}}^p\right)^{\frac{1}{p}}}\right)^{2(p-1)}\right. \\ -(p-1)a(d+1)\left(1+a\|M(s)\|_{\mathrm{W}^{1,p}}^p\right)^{\frac{1}{p}-2}\sum_{j\in\mathbb{N}} q\left(M(s),U(t-s)\xi(s)\Phi e_j\right)^2 \\ \left.(p-1)\|M(s)\|_{\mathrm{W}^{1,p}}^{p-2}\left(1+a\|M(s)\|_{\mathrm{W}^{1,p}}^p\right)^{\frac{1}{p}-1}\sum_{j\in\mathbb{N}}\|U(t-s)\xi(s)\Phi e_j\|_{\mathrm{W}^{1,p}}^2\right]ds. \end{array}$$

Since the term in parenthesis in the first part is an increasing function of $\|M(s)\|_{\mathrm{W}^{1,p}}$, the second term is non positive and

$$\begin{array}{l} \|M(s)\|_{\mathrm{W}^{1,p}}^{p-2}\left(1+a\|M(s)\|_{\mathrm{W}^{1,p}}^p\right)^{\frac{1}{p}-1} \\ = a^{\frac{2}{p}-1}\left(a\|M(s)\|_{\mathrm{W}^{1,p}}^p\right)^{1-\frac{2}{p}}\left(1+a\|M(s)\|_{\mathrm{W}^{1,p}}^p\right)^{\frac{1}{p}-1} \\ \le a^{\frac{2}{p}-1}\left(1+a\|M(s)\|_{\mathrm{W}^{1,p}}^p\right)^{1-\frac{2}{p}}\left(1+a\|M(s)\|_{\mathrm{W}^{1,p}}^p\right)^{\frac{1}{p}-1} \le a^{\frac{2}{p}-1}, \end{array}$$

we obtain that

$$R_a(t_0) \le (d+1)(d+p)a^{\frac{2}{p}}\int_0^{t_0}\sum_{j\in\mathbb{N}}\|U(t-s)\xi(s)\Phi e_j\|_{\mathrm{W}^{1,p}}^2 ds.$$

Finally, from (B.2.1), Hölder's inequality and the Sobolev injections we obtain that for any t_0 in $[0,T]$,

$$R_a(t_0) \le \frac{2(d+1)(d+p)a^{\frac{2}{p}}}{2}c\left(\frac{r(p)d}{2}\right)^2\|\Phi\|_{\mathcal{L}_2^{0,s}}^2\eta\int_0^T |t-s|^{-\frac{4}{r(p)}}ds,$$

the integral is finite since $p<\frac{2d}{d-1}$ and thus $R_a(t_0)\le\frac{\kappa(\eta)a^{\frac{2}{p}}}{4}$. Also, since $\left(\exp\left(E_a(t_0)\right)\right)_{t_0\in[0,T]}$ is a martingale, the Novikov condition is satisfied from the above, and from Doob's inequality, we obtain

$$\begin{array}{l} \mathbb{P}\left(\sup_{t_0\in[0,T]}\left\|\int_0^{t_0}U(t-s)\xi(s)dW(s)\right\|_{\mathrm{W}^{1,p}}\ge\delta\right) \\ = \mathbb{P}\left(\sup_{t_0\in[0,T]}\exp\left(g_a(M(t_0))\right)\ge\exp\left(\left(1+a\delta^p\right)^{\frac{1}{p}}\right)\right) \\ \le \mathbb{P}\left(\sup_{t_0\in[0,T]}\exp\left(E_a(t_0)\right)\ge\exp\left(\left(1+a\delta^p\right)^{\frac{1}{p}}-1-\frac{\kappa(\eta)a^{\frac{2}{p}}}{4}\right)\right) \end{array}$$

$$\begin{aligned}&\le \exp\left(-(1+a\delta^p)^{\frac{1}{p}}+1+\tfrac{\kappa(\eta)a^{\frac{2}{p}}}{4}\right)\\&\le e\exp\left(-a^{\frac{1}{p}}\delta+\tfrac{\kappa(\eta)a^{\frac{2}{p}}}{4}\right).\end{aligned}$$

The last inequality holds for arbitrary a positive. Minimizing on a, one finally obtains the desired estimate. □

Proposition B.3.3 (Exponential tail estimates) *If Z, defined by $Z(t)=\int_0^t U(t-s)\xi(s)dW(s)$, is such that there exists η positive such that $\|\xi\|^2_{\mathrm{C}([0,T];\mathrm{H}^1)}\le \eta$ a.s., then for any p in $\tilde{\mathcal{A}}(d)$, T and δ positive,*

$$\begin{aligned}&\mathbb{P}\left(\|Z\|_{\mathrm{C}([0,T];\mathrm{H}^1)}\ge\delta\right)\le 3\exp\left(-\tfrac{\delta^2}{\kappa_1(\eta)}\right)\\&\mathbb{P}\left(\|Z\|_{\mathrm{L}^{r(p)}(0,T;W^{1,p})}\ge\delta\right)\le c\exp\left(-\tfrac{\delta^2}{\kappa_2(\eta)}\right)\end{aligned}$$

where $c=2e+\exp\left((2ek_0!)^{\frac{1}{k_0}}\right)$, $k_0=2\vee\min\{k\in\mathbb{N}:2k\ge r(p)\}$

$$\kappa_1(\eta)=T4c(\infty)^2\|\Phi\|^2_{\mathcal{L}_2^{0,s}}\eta,$$

$$\kappa_2(\eta)=\frac{8c\left(\frac{r(p)d}{2}\right)^2T^{1-\frac{2}{r(p)}}(d+1)(d+p)\|\Phi\|^2_{\mathcal{L}_2^{0,s}}}{1-\frac{4}{r(p)}}\eta,$$

$c\left(\frac{r(p)d}{2}\right)$ and $c(\infty)$ are the norms of the continuous embeddings $\mathrm{H}^s_{\mathbb{R}}\subset\mathrm{W}^{1,\frac{r(p)d}{2}}_{\mathbb{R}}$ and $\mathrm{H}^s_{\mathbb{R}}\subset\mathrm{W}^{1,\infty}_{\mathbb{R}}$.

Proof. The first estimate. We recall that in H^1 we may write, using the series expansion of the Wiener process and that $(U_t)_{t\in\mathbb{R}}$ is a unitary group, see [81], $Z(t)=U(t)\int_0^t U(-s)\xi(s)dW(s)$. Since $U(-s)$ is an isometry, one obtains that for every s in $[0,T]$,

$$\begin{aligned}\|U(-s)\xi(s)\Phi\|_{\mathcal{L}_2(\mathrm{L}^2,\mathrm{H}^1)}&\le\|L_s\|_{\mathcal{L}_c\left(\mathrm{H}^s_{\mathbb{R}},\mathrm{H}^1\right)}\|\Phi\|_{\mathcal{L}_2^{0,s}}\\&\le c(\infty)\|L_s\|_{\mathcal{L}_c\left(\mathrm{W}^{1,\infty}_{\mathbb{R}},\mathrm{H}^1\right)}\|\Phi\|_{\mathcal{L}_2^{0,s}}\\&\le c(\infty)\|\xi(s)\|_{\mathrm{H}^1}\|\Phi\|_{\mathcal{L}_2^{0,s}}\end{aligned}$$

where L is such that $L_su=\xi(s)u$. Consequently, we obtain that

$$\int_0^T\|U(-s)\xi(s)\Phi\|^2_{\mathcal{L}_2^{0,1}}ds\le c(\infty)^2\|\Phi\|^2_{\mathcal{L}_2^{0,s}}T\|\xi(s)\|^2_{\mathrm{C}([0,T],\mathrm{H}^1)}ds$$

We conclude using [117][Theorem 2.1]. The result still holds when the operator takes its value in another Hilbert space.

The second estimate. From Markov's inequality it is enough to show that

$$\mathbb{E}\left[\exp\left(\frac{1}{\kappa_2(\eta)}\left\|\int_0^{\cdot} U(\cdot-s)\xi(s)dW(s)\right\|^2_{\mathrm{L}^{r(p)}(0,T;\mathrm{W}^{1,p})}\right)\right]\leq c.$$

For $k\geq k_0$, Jensen's inequality along with Fubini's theorem, Lemma B.3.2, the change of variables and the integration by parts formulae give that

$$\begin{aligned}
&\mathbb{E}\left[\left(\tfrac{1}{\sqrt{\kappa_2(\eta)}}\left\|\int_0^{\cdot} U(\cdot-s)\xi(s)dW(s)\right\|_{\mathrm{L}^{r(p)}(0,T;W^{1,p})}\right)^{2k}\right]\\
&\leq \tfrac{1}{T}\int_0^T \mathbb{E}\left[\left(\tfrac{T}{(\kappa_2(\eta))^{\frac{r(p)}{2}}}\left\|\int_0^t U(t-s)\xi(s)dW(s)\right\|^{r(p)}_{\mathrm{W}^{1,p}}\right)^{\frac{2k}{r(p)}}\right]dt\\
&\leq \tfrac{1}{T}\int_0^T\int_0^\infty \mathbb{P}\left(\left\|\int_0^t U(t-s)\xi(s)dW(s)\right\|_{\mathrm{W}^{1,p}}\geq\left(\tfrac{(\kappa_2(\eta))^{\frac{r(p)}{2}}}{T}\right)^{\frac{1}{r(p)}}u^{\frac{1}{2k}}\right)dudt\\
&\leq \tfrac{1}{T}\int_0^T\int_0^\infty e\exp\left(-\tfrac{\kappa_2(\eta)}{T^{\frac{2}{r(p)}}}\tfrac{u^{\frac{1}{k}}}{\kappa(\eta)}\right)dudt\\
&\leq e\int_0^\infty \exp\left(-2u^{\frac{1}{k}}\right)du = e\int_0^\infty kv^{k-1}\exp(-2v)dv = 2e\int_0^\infty v^k\exp(-2v)dv.
\end{aligned}$$

Thus, using Fubini's theorem one obtains

$$\begin{aligned}
&\mathbb{E}\left[\left(\tfrac{1}{\sqrt{\kappa_2(\eta)}}\left\|\int_0^{\cdot} U(\cdot-s)\xi(s)dW(s)\right\|_{\mathrm{L}^{r(p)}(0,T;W^{1,p})}\right)^{2k_0}\right]\\
&\leq k_0!\sum_{k\geq k_0}\tfrac{1}{k!}\mathbb{E}\left[\left(\tfrac{1}{\sqrt{\kappa_2(\eta)}}\left\|\int_0^{\cdot} U(\cdot-s)\xi(s)dW(s)\right\|_{\mathrm{L}^{r(p)}(0,T;W^{1,p})}\right)^{2k}\right]\\
&\leq k_0!\sum_{k\geq k_0}\tfrac{1}{k!}2e\int_0^\infty v^k\exp(-2v)dv\\
&\leq k_0!\sum_{k\in\mathbb{N}}\tfrac{1}{k!}2e\int_0^\infty v^k\exp(-2v)dv = 2ek_0!,
\end{aligned}$$

hence from Hölder's inequality

$$\begin{aligned}
&\sum_{k=0}^{k_0-1}\tfrac{1}{k!}\mathbb{E}\left[\left(\tfrac{1}{\sqrt{\kappa_2(\eta)}}\left\|\int_0^{\cdot} U(\cdot-s)\xi(s)dW(s)\right\|_{\mathrm{L}^{r(p)}(0,T;W^{1,p})}\right)^{2k}\right]\\
&=\sum_{k=0}^{k_0-1}\tfrac{1}{\kappa_2(\eta)^k k!}\mathbb{E}\left[\left\{\left(\left\|\int_0^{\cdot} U(\cdot-s)\xi(s)dW(s)\right\|_{\mathrm{L}^{r(p)}(0,T;W^{1,p})}\right)^{2k_0}\right\}^{\frac{k}{k_0}}\right]\\
&\leq\sum_{k=0}^{k_0-1}\tfrac{1}{\kappa_2(\eta)^k k!}\mathbb{E}\left[\left(\left\|\int_0^{\cdot} U(\cdot-s)\xi(s)dW(s)\right\|_{\mathrm{L}^{r(p)}(0,T;W^{1,p})}\right)^{2k_0}\right]^{\frac{k}{k_0}}\\
&\leq\sum_{k=0}^{k_0-1}\frac{\left[(2ek_0!)^{\frac{1}{k_0}}\right]^k}{k!}\leq\exp\left((2ek_0!)^{\frac{1}{k_0}}\right).
\end{aligned}$$

The end of the proof is now straightforward. $\square$

B.4 Almost continuity of the Itô map

Proposition B.4.1 *For every positive a, R and ρ, u_0 in H^1, f in C_a, $T < \mathcal{T}(\mathbf{S}(u_0, f))$, p in $\mathcal{A}(d)$, there exists positive ϵ_0, γ and r such that for every ϵ in $(0, \epsilon_0]$ and $\tilde{u}_0$ in $B_{\mathrm{H}^1}(u_0, r)$,*

$$\epsilon \log \mathbb{P}\left(\left\|u^{\epsilon,\tilde{u}_0} - \mathbf{S}(u_0, f)\right\|_{X^{(T,p)}} \geq \rho; \left\|\sqrt{\epsilon}W - f\right\|_{\mathrm{C}\left([0,T];\mathrm{H}^s_{\mathbb{R}}\right)} < \gamma\right) \leq -R.$$

Proof. Take u_0 in H^1, f in C_a, a, R and ρ positive, $T < \mathcal{T}(\mathbf{S}(u_0, f))$ and p in $\mathcal{A}(d)$.

Step 1: Change of measure to center the Wiener process around f. The function f in C_a is such that there exists h in $\mathrm{L}^2\left(0, T; \mathrm{L}^2\right)$ such that $f(\cdot) = \int_0^{\cdot} \Phi h(s)ds$ and $\frac{1}{2}\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \leq a$. We denote by W^ϵ the process

$$W(t) - \frac{1}{\sqrt{\epsilon}}\int_0^t \frac{\partial f}{\partial s} ds = W(t) - \frac{1}{\sqrt{\epsilon}}\int_0^t \Phi h(s)ds = \Phi\left(W_c(t) - \frac{1}{\sqrt{\epsilon}}\int_0^t h(s)ds\right).$$

Since $\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \|h(s)\|^2_{\mathrm{L}^2}\right) ds\right] < \infty$, the Novikov condition is satisfied and the Girsanov theorem gives that W^ϵ is a $\mu-$Wiener process on $\mathrm{C}\left([0, T]; \mathrm{H}^s_{\mathbb{R}}\right)$ under the probability $\mathbb{P}^\epsilon$ defined by

$$\left.\frac{\mathrm{d}\mathbb{P}^\epsilon}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left(\frac{1}{\sqrt{\epsilon}}\int_0^t (h, dW_c(s))_{\mathrm{L}^2} - \frac{1}{2\epsilon}\int_0^T \|h(s)\|^2_{\mathrm{L}^2} ds\right)$$

Set $u^\epsilon(t) = \exp\left(-\frac{1}{\sqrt{\epsilon}}\int_0^t (h, dW_c(s))_{\mathrm{L}^2}\right)$ and λ such that $a - \lambda < -R$ and

$$A = \left\{\left\|u^{\epsilon,\tilde{u}_0} - \mathbf{S}(u_0, f)\right\|_{X^{(T,p)}} \geq \rho; \left\|\sqrt{\epsilon}W - f\right\|_{\mathrm{C}\left([0,T];\mathrm{H}^s_{\mathbb{R}}\right)} < \gamma\right\},$$

then since $\left(\exp\left(-\frac{1}{\sqrt{\epsilon}}\int_0^t (h, dW_c(s))_{\mathrm{L}^2} - \frac{1}{2\epsilon}\int_0^t \|h(s)\|^2_{\mathrm{L}^2} ds\right)\right)_{t\in[0,T]}$ is a uniformly integrable martingale

$$\begin{aligned}
\mathbb{P}(A) &\leq \mathbb{E}_{\mathbb{P}_\epsilon}\left\{\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}_\epsilon}\mathbb{1}_{A\cap\left\{u^\epsilon(T)\leq\exp\left(\frac{\lambda}{\epsilon}\right)\right\}}\right\} + \mathbb{P}\left(u^\epsilon(T) > \exp\left(\frac{\lambda}{\epsilon}\right)\right) \\
&\leq \mathbb{E}_{\mathbb{P}_\epsilon}\left\{\mathbb{1}_A \exp\left(\frac{\lambda}{\epsilon} + \frac{1}{2\epsilon}\int_0^T \|h(s)\|^2_{\mathrm{L}^2} ds\right)\right\} + \exp\left(-\frac{\lambda}{\epsilon}\right)\mathbb{E}\left(u^\epsilon(T)\right) \\
&\leq \exp\left(\frac{\lambda+a}{\epsilon}\right)\mathbb{P}_\epsilon(A) + \exp\left(\frac{a-\lambda}{\epsilon}\right).
\end{aligned}$$

Finally it is enough to prove that there exists positive ϵ_0, γ and r such that for every ϵ in $(0,\epsilon_0]$ and $\tilde{u}_0$ in $B_{\mathrm{H}^1}(u_0,r)$, $\epsilon\log\mathbb{P}_\epsilon(A)\le -R-\lambda-a$, or equivalently that

$$\epsilon\log\mathbb{P}_\epsilon\left(\left\|v^{\epsilon,\tilde{u}_0}-\mathbf{S}(u_0,f)\right\|_{X^{(T,p)}}\ge\rho;\left\|\sqrt{\epsilon}W_\epsilon\right\|_{C\left([0,T];\mathrm{H}^s_{\mathbb{R}}\right)}<\gamma\right)\le -R-\lambda-a,$$

where $v^{\epsilon,\tilde{u}_0}$ satisfies $v^{\epsilon,\tilde{u}_0}(0)=\tilde{u}_0$ and

$$idv^{\epsilon,\tilde{u}_0}=\left(\Delta v^{\epsilon,\tilde{u}_0}+\lambda|v^{\epsilon,\tilde{u}_0}|^{2\sigma}v^{\epsilon,\tilde{u}_0}+\frac{\partial f}{\partial t}v^{\epsilon,\tilde{u}_0}-\frac{i\epsilon}{2}F_\Phi v^{\epsilon,\tilde{u}_0}\right)dt+\sqrt{\epsilon}v^{\epsilon,\tilde{u}_0}dW_\epsilon.$$

Step 2: Reduction to estimates for the stochastic convolution. Note, this is standard fact, that the unboundedness of the drift and coefficient of the Wiener process is not a limitation since the result of Proposition B.4.1 is local. A localisation argument is therefore used to overcome the apparent difficulty. We replace T by

$$\tau_\rho=\inf\left\{t:\ \|v^{\epsilon,\tilde{u}_0}-\mathbf{S}(u_0,f)\|_{X^{(t,p)}}\ge\rho\right\}\wedge T.$$

Since $T<\mathcal{T}(\mathbf{S}(u_0,f))$, $v^{\epsilon,\tilde{u}_0}$ satisfies

$$\|v^{\epsilon,\tilde{u}_0}\|_{X^{(\tau_\rho,p)}}\le\rho+\|\mathbf{S}(u_0,f)\|_{X^{(\tau_\rho,p)}}=D.$$

With computations similar to that of the proofs of Proposition B.3.1 herein and of Theorem 4.1 in [37] with a cutt-off function in front of the nonlinearity of the form $\theta\left(\frac{\|\mathbf{S}(u_0,f)\|_{X^{(s,p)}}}{D}\right)$ and $\theta\left(\frac{\|v^{\epsilon,\tilde{u}_0}\|_{X^{(s,p)}}}{D}\right)$, we obtain for t positive and $\nu\in\left(0,1-\frac{\sigma(d-2)}{2}\right)$,

$$\begin{aligned}&\left\|v^{\epsilon,\tilde{u}_0}-\mathbf{S}(u_0,f)\right\|_{X^{(t\wedge\tau_\rho,p)}}\le C\|\tilde{u}_0-u_0\|_{\mathrm{H}^1}+\sqrt{\epsilon}\left\|\int_0^\cdot U(\cdot-s)v^{\epsilon,\tilde{u}_0}(s)dW_\epsilon(s)\right\|_{X^{(t\wedge\tau_\rho,p)}}\\&+C\left[(t\wedge\tau_\rho)^\nu(D^{2\sigma})(1+D)+(t\wedge\tau_\rho)^{\frac12-\frac{1}{r(p)}}\sqrt{a}+\epsilon(t\wedge\tau_\rho)^{1-\frac{2}{r(p)}}\right]\left\|v^{\epsilon,\tilde{u}_0}-\mathbf{S}(u_0,f)\right\|_{X^{(t\wedge\tau_\rho,p)}}\\&+C\epsilon(t\wedge\tau_\rho)^{1-\frac{2}{r(p)}}\|\mathbf{S}(u_0,f)\|_{X^{(t\wedge\tau_\rho,p)}}.\end{aligned}$$

Set $\epsilon\le 1$, then for $t=t^*$ small enough we obtain

$$\begin{aligned}\left\|v^{\epsilon,\tilde{u}_0}-\mathbf{S}(u_0,f)\right\|_{X^{(t^*\wedge\tau_\rho,p)}}\le\ &2\left(C\|\tilde{u}_0-u_0\|_{\mathrm{H}^1}+C\epsilon D\right.\\&\left.+\sqrt{\epsilon}\left\|\int_0^\cdot U(\cdot-s)v^{\epsilon,\tilde{u}_0}(s)dW_\epsilon(s)\right\|_{X^{(\tau_\rho,p)}}\right).\end{aligned}\tag{B.4.1}$$

Set $N=\left\lfloor\frac{\tau_\rho}{t^*\wedge\tau_\rho}\right\rfloor$, and for i in $\{0,...,N\}$, $T_i=it^*$ and $T_{N+1}=T$. Inequality (B.4.1) also holds for $\left\|v^{\epsilon,\tilde{u}_0}-\mathbf{S}(u_0,f)\right\|_{X^{(T_i,T_{i+1},p)}}$ for every i in $\{0,...,N\}$, replacing $\|\tilde{u}_0 - u_0\|_{\mathrm{H}^1}$ by $\left\|v^{\epsilon,\tilde{u}_0}(T_i) - \mathbf{S}(u_0,f)(T_i)\right\|_{\mathrm{H}^1}$.

As for i in $\{1,...,N\}$, $\left\|v^{\epsilon,\tilde{u}_0}(T_i) - \mathbf{S}(u_0,f)(T_i)\right\|_{\mathrm{H}^1} \leq \left\|v^{\epsilon,\tilde{u}_0} - \mathbf{S}(u_0,f)\right\|_{X^{(T_{i-1},T_i,p)}}$, we obtain using the triangle inequality that

$$\begin{aligned}&\left\|v^{\epsilon,\tilde{u}_0} - \mathbf{S}(u_0,f)\right\|_{X^{(\tau_\rho,p)}}\\ &\leq 2(N+1)\left(\sqrt{\epsilon}\left\|\int_0^\cdot U(\cdot - s)v^{\epsilon,\tilde{u}_0}(s)dW_\epsilon(s)\right\|_{X^{(\tau_\rho,p)}} + C\epsilon D\right)\\ &\ +2C\sum_{i=1}^{N-1}\left\|v^{\epsilon,\tilde{u}_0} - \mathbf{S}(u_0,f)\right\|_{X^{(T_{i-1},T_i,p)}} + 2C\|u_0 - \tilde{u}_0\|_{\mathrm{H}^1}\\ &\leq 2(N+1)\left(\sum_{i=0}^{N-1}(2C)^i\right)\left(\sqrt{\epsilon}\left\|\int_0^\cdot U(\cdot - s)v^{\epsilon,\tilde{u}_0}(s)dW_\epsilon(s)\right\|_{X^{(\tau_\rho,p)}} + C\epsilon D\right)\\ &\ +(2C)^N\|u_0 - \tilde{u}_0\|_{\mathrm{H}^1}.\end{aligned}$$

We may choose $2C > 1$, then it is enough to show that there exists positive ϵ_0, γ and r such that $(2C)^N r < \rho$ and for every ϵ in $(0,\epsilon_0]$ and $\tilde{u}_0$ in $B_{\mathrm{H}^1}(u_0,r)$,

$$\begin{aligned}\epsilon\log\mathbb{P}_\epsilon\Big(\sqrt{\epsilon}\left\|\int_0^\cdot U(\cdot - s)v^{\epsilon,\tilde{u}_0}(s)dW_\epsilon(s)\right\|_{X^{(\tau_\rho,p)}} + C\epsilon D \geq \tfrac{(2C-1)(\rho-(2C)^N r)}{2(N+1)((2C)^N-1)};\\ \|\sqrt{\epsilon}W_\epsilon\|_{\mathrm{C}\left([0,T];\mathrm{H}^s_{\mathbb{R}}\right)} < \gamma\Big) \leq -R - \lambda - a.\end{aligned}$$

Step 3: The case of the stochastic convolution. We now need that for u_0 in H^1, f in C_a, a, R and ρ positive, $T < \mathcal{T}(\mathbf{S}(u_0,f))$ and p in $\mathcal{A}(d)$, there exists ϵ_0, γ and r positive such that for every ϵ in $(0,\epsilon_0]$ and $\tilde{u}_0$ in $B_{\mathrm{H}^1}(u_0,r)$,

$$\begin{aligned}&\epsilon\log\mathbb{P}_\epsilon\left(\sqrt{\epsilon}\left\|\int_0^\cdot U(\cdot - s)v^{\epsilon,\tilde{u}_0}(s)dW_\epsilon(s)\right\|_{X^{(\tau_\rho,p)}} \geq \rho; \|\sqrt{\epsilon}W_\epsilon\|_{\mathrm{C}\left([0,\tau_\rho];\mathrm{H}^s_{\mathbb{R}}\right)} < \gamma\right)\\ &\leq -R.\end{aligned}$$

For n in $\mathbb{N}$ and i in $\{0,...,n\}$, we set $t_i = \frac{i\tau_\rho}{n}$ and the same approximation as (B.3.1)

$$v^{\epsilon,\tilde{u}_0,n}(t) = U(t - t_i)\left(v^{\epsilon,\tilde{u}_0}(t_i)\right), \text{ for } t_i \leq t < t_{i+1}.$$

For any δ positive we may write

$$\begin{aligned}&\mathbb{P}_\epsilon\left(\sqrt{\epsilon}\left\|\int_0^\cdot U(\cdot - s)v^{\epsilon,\tilde{u}_0}(s)dW_\epsilon(s)\right\|_{X^{(\tau_\rho,p)}} \geq \rho; \|\sqrt{\epsilon}W_\epsilon\|_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^s_{\mathbb{R}}} < \gamma\right)\\ &\leq \mathbb{P}_1 + \mathbb{P}_2 + \mathbb{P}_3,\end{aligned}$$

where

$$\begin{aligned}\mathbb{P}_1 = \mathbb{P}_\epsilon\Big(\sqrt{\epsilon}\left\|\int_0^\cdot U(\cdot - s)\left(v^{\epsilon,\tilde{u}_0}(s) - v^{\epsilon,\tilde{u}_0,n}(s)\right)dW_\epsilon(s)\right\|_{X^{(\tau_\rho,p)}} \geq \tfrac{\rho}{2};\\ \left\|v^{\epsilon,\tilde{u}_0} - v^{\epsilon,\tilde{u}_0,n}\right\|_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^1)} < \delta\Big),\end{aligned}$$

$$\mathbb{P}_2 = \mathbb{P}_\epsilon\left(\left\|v^{\epsilon,\tilde{u}_0} - v^{\epsilon,\tilde{u}_0,n}\right\|_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^1)} \geq \delta\right),$$

$$\mathbb{P}_3 = \mathbb{P}_\epsilon \Big(\sqrt{\epsilon} \left\| \int_0^\cdot U(\cdot - s) v^{\epsilon,\tilde{u}_0,n}(s) dW_\epsilon(s) \right\|_{X^{(\tau_\rho,p)}} \geq \tfrac{\rho}{2}; \|\sqrt{\epsilon} W_\epsilon\|_{\mathrm{C}\left([0,\tau_\rho];\mathrm{H}^s_{\mathbb{R}}\right)} < \gamma; \\ \left\| v^{\epsilon,\tilde{u}_0} - v^{\epsilon,\tilde{u}_0,n} \right\|_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^1)} < \delta \Big).$$

From Proposition B.3.3,

$$\mathbb{P}_1 \leq C \exp \left(-\frac{\rho^2}{4\epsilon \left(\kappa_1(\delta^2) \vee \kappa_2(\delta^2)\right)} \right),$$

thus for any $\epsilon < 1$ and δ small enough, $\mathbb{P}_1 < -R - 1$.

Also $\mathbb{P}_2 \leq \mathbb{P}_{21} + \mathbb{P}_{22}$ where

$$\mathbb{P}_{21} = \mathbb{P}_\epsilon \left(\sup_{i \in \{0,\dots,n-1\}} \sqrt{\epsilon} \left\| \int_{t_i}^\cdot U(t-s) v^{\epsilon,\tilde{u}_0}(s) dW_\epsilon(s) \right\|_{\mathrm{C}([t_i,t_{i+1}];\mathrm{H}^1)} \geq \frac{\delta}{2} \right)$$

and $\mathbb{P}_{22}$ equals

$$\mathbb{P}_\epsilon \left(\sup_{i \in \{0,\dots,n-1\}} \left\| \int_{t_i}^\cdot U(t-s) \left[\lambda \left| v^{\epsilon,\tilde{u}_0}(s) \right|^{2\sigma} + \frac{\partial f}{\partial s}(s) - \frac{i\epsilon}{2} F_\Phi \right] v^{\epsilon,\tilde{u}_0}(s) ds \right\|_{\mathrm{C}([t_i,t_{i+1}];\mathrm{H}^1)} \geq \frac{\delta}{2} \right).$$

From Proposition B.3.3, $\mathbb{P}_{21} \leq 3n \exp\left(-\frac{C\delta^2 n}{\tau_\rho D^2} \right)$ and $\mathbb{P}_{22} = 0$ for n large enough. Indeed, with calculations similar to that of the proof of Theorem 4.1 in [37] and of Proposition B.3.1, we obtain for $\epsilon < 1$,

$$\sup_{i \in \{0,\dots,n-1\}} \left\| \int_{t_i}^\cdot U(t-s) \left[\lambda \left| v^{\epsilon,\tilde{u}_0}(s) \right|^{2\sigma} + \tfrac{\partial f}{\partial s}(s) - \tfrac{i\epsilon}{2} F_\Phi \right] v^{\epsilon,\tilde{u}_0}(s) ds \right\|_{\mathrm{C}([t_i,t_{i+1}];\mathrm{H}^1)} \\ \leq C \left[\left(\tfrac{\tau_\rho}{n}\right)^\nu D^{2\sigma+1} + \left(\tfrac{\tau_\rho}{n}\right)^{\frac{1}{2} - \frac{1}{r(p)}} D\sqrt{a} + \tfrac{1}{2} \left(\tfrac{\tau_\rho}{n}\right)^{1 - \frac{2}{r(p)}} D \right],$$

then, for every δ positive and $0 < \epsilon < 1$, for n large enough $\epsilon \log \mathbb{P}_2 < -R-1$.

So far we have obtained that for δ small enough and n large enough, for any $0 < \epsilon < \frac{1}{2\log(2)}$, $\epsilon \log\left(\mathbb{P}_1 + \mathbb{P}_2\right) \leq -R - \frac{1}{2}$.

Now fix δ as above. Note that the condition $\left\| v^{\epsilon,\tilde{u}_0} - v^{\epsilon,\tilde{u}_0,n} \right\|_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^1)} < \delta$ in $\mathbb{P}_3$ implies that $\left\| v^{\epsilon,\tilde{u}_0,n} \right\|_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^1)} < D + \delta$.

Set $\underline{t} = \max\{t_i : t_i \leq t,\ i \in \{0,\dots,n\}\}$ and

$$E = \left\{ \left\| \sqrt{\epsilon} W_\epsilon \right\|_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^s_{\mathbb{R}})} < \gamma \ ; \ \left\| v^{\epsilon,\tilde{u}_0} - v^{\epsilon,\tilde{u}_0,n} \right\|_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^1)} < \delta \right\}.$$

As $p < \frac{2(3d-1)}{3(d-1)}$ there exists $p < \tilde{p} < \frac{2d}{d-1}$ and η positive such that $1 - \frac{p-2}{\tilde{p}-2}\left(1 + \frac{2}{r(\tilde{p})} + \eta\right)$ is positive. Thus, from Hölder's inequality, for $\theta = \frac{\tilde{p}-p}{\tilde{p}-2}$,

$$\mathbb{P}_3 \leq \mathbb{P}_{31} + \mathbb{P}_{32} + \mathbb{P}_{33} + \mathbb{P}_{34} + \mathbb{P}_{35},$$

where

$$\mathbb{P}_{31} = \mathbb{P}_\epsilon\left(\sqrt{\epsilon}\left\|\int_{\cdot}^{\cdot} U(\cdot-s)v^{\epsilon,\tilde{u}_0,n}(s)dW_\epsilon(s)\right\|_{\mathrm{L}^{r(\tilde{p})}(0,\tau_\rho;\mathrm{W}^{1,\tilde{p}})} \geq n^{\left(1+\frac{2}{r(\tilde{p})}+\eta\right)\frac{1}{2}}; \right.$$
$$\left. \left\|v^{\epsilon,\tilde{u}_0} - v^{\epsilon,\tilde{u}_0,n}\right\|_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^1)} < \delta\right),$$

$$\mathbb{P}_{32} = \mathbb{P}_\epsilon\left(\sqrt{\epsilon}\left\|\int_{\cdot}^{\cdot} U(\cdot-s)v^{\epsilon,\tilde{u}_0,n}(s)dW_\epsilon(s)\right\|^\theta_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^1)} n^{\left(1+\frac{2}{r(\tilde{p})}+\eta\right)\frac{1-\theta}{2}} \geq \frac{\rho}{8}; \right.$$
$$\left. \left\|v^{\epsilon,\tilde{u}_0} - v^{\epsilon,\tilde{u}_0,n}\right\|_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^1)} < \delta\right),$$

$$\mathbb{P}_{33} = \mathbb{P}_\epsilon\left(\sqrt{\epsilon}\left\|\int_{\cdot}^{\cdot} U(\cdot-s)v^{\epsilon,\tilde{u}_0,n}(s)dW_\epsilon(s)\right\|_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^1)} \geq \frac{\rho}{8}; \right.$$
$$\left. \left\|v^{\epsilon,\tilde{u}_0} - v^{\epsilon,\tilde{u}_0,n}\right\|_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^1)} < \delta\right),$$

$$\mathbb{P}_{34} = \mathbb{P}_\epsilon\left(\sqrt{\epsilon}\left\|\int_0^{\cdot} U(\cdot-s)v^{\epsilon,\tilde{u}_0,n}(s)dW_\epsilon(s)\right\|_{\mathrm{L}^{r(p)}(0,\tau_\rho;W^{1,p})} \geq \frac{\rho}{8}; E\right),$$

$$\mathbb{P}_{35} = \mathbb{P}_\epsilon\left(\sqrt{\epsilon}\left\|\int_0^{\cdot} U(\cdot-s)v^{\epsilon,\tilde{u}_0,n}(s)dW_\epsilon(s)\right\|_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^1)} \geq \frac{\rho}{8}; E\right).$$

The probability $\mathbb{P}_{31}$ is less than

$$\sum_{i=0}^{n-1} \mathbb{P}_\epsilon\left(\sqrt{\epsilon}\left\|\int_{t_i}^{t_{i+1}} U(\cdot-s)v^{\epsilon,\tilde{u}_0,n}(s)dW_\epsilon(s)\right\|_{\mathrm{L}^{r(\tilde{p})}(t_i,t_{i+1};W^{1,\tilde{p}})} \geq n^{\left(-1+\frac{2}{r(\tilde{p})}+\eta\right)\frac{1}{2}}; \right.$$
$$\left. \left\|v^{\epsilon,\tilde{u}_0} - v^{\epsilon,\tilde{u}_0,n}\right\|_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^1)} < \delta\right)$$

which is itself, from Proposition B.3.3, less than

$$nC\exp\left(-\frac{n^{-1+\frac{2}{r(\tilde{p})}+\eta}}{C\left(\frac{\tau_\rho}{n}\right)^{1-\frac{2}{r(\tilde{p})}}\epsilon(\delta+D)^2}\right).$$

Thus, for n large enough, $\epsilon \leq 1$ and δ positive, $\epsilon \log \mathbb{P}_{31} < -R-1$.

The first exponential tail estimate of Proposition B.3.3 gives that

$$\mathbb{P}_{32} \leq nC\exp\left(-\frac{\rho^2}{Cn^{\left(1+\frac{2}{r(\tilde{p})}+\eta\right)(1-\theta)}\left(\frac{\tau_\rho}{n}\right)\epsilon(\delta+D)^2}\right),$$

and, from the choice of $\tilde{p}$, for n large enough, for any ϵ and δ positive, $\epsilon \log \mathbb{P}_{32} < -R-1$.

The same holds for $\mathbb{P}_{33}$ even more clearly.

The decay estimate (B.2.1) along with Hölder's inequality give that the mapping $w \mapsto U(t-t_j)v^{\epsilon,\tilde{u}_0}(t_j)w$ from $\mathrm{H}^s_{\mathbb{R}}$ into $\mathrm{W}^{1,p}$ is continuous. Thus, we may write

$$\begin{aligned}
&\left\|\int_0^{\cdot} U(\cdot-s)v^{\epsilon,\tilde{u}_0,n}(s)dW_\epsilon(s)\right\|_{\mathrm{L}^{r(p)}(0,T;W^{1,p})}\\
&=\left\|\sum_{i=1}^{n-1}\mathbb{1}_{t_i\le t<t_{i+1}}\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}U(t-t_j)v^{\epsilon,\tilde{u}_0}(t_j)dW_\epsilon(s)\right\|_{\mathrm{L}^{r(p)}(0,T;W^{1,p})}\\
&\le\sum_{i=1}^{n-1}\sum_{j=0}^{i-1}\left\|\int_{t_j}^{t_{j+1}}U(t-t_j)v^{\epsilon,\tilde{u}_0}(t_j)dW_\epsilon(s)\right\|_{\mathrm{L}^{r(p)}(t_i,t_{i+1};W^{1,p})}\\
&\le C\left(\frac{(n-1)(n-2)}{2}\right)\left(\frac{\tau_\rho}{n}\right)^{-\frac{2}{r(p)}}D\gamma,
\end{aligned}$$

and obtain that, for any n in $\mathbb{N}$, for γ small enough $\mathbb{P}_{34}=0$.

Similarly we write, using the continuity of the group and Hölder's inequality,

$$\begin{aligned}
&\left\|\int_0^{\cdot} U(\cdot-s)v^{\epsilon,\tilde{u}_0,n}(s)dW_\epsilon(s)\right\|_{\mathrm{C}(0,T;\mathrm{H}^1)}\\
&=\max_{i=1,\dots,n-1}\left\|\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}U(t_i-t_j)v^{\epsilon,\tilde{u}_0}(t_j)dW_\epsilon(s)\right\|_{\mathrm{H}^1}\\
&\le\sum_{j=0}^{n-1}\|v^{\epsilon,\tilde{u}_0}(t_j)\|_{\mathrm{H}^1}\|W_\epsilon(t_{j+1})-W_\epsilon(t_j)\|_{\mathrm{H}^s_{\mathbb{R}}}\\
&\le 2nD\gamma.
\end{aligned}$$

Thus, for any n in $\mathbb{N}$, for γ small enough $\mathbb{P}_{35}=0$.

Finally, when δ is fixed, for n large enough and a particular choice of γ depending on n and δ, we obtain that for any $0<\epsilon<\frac{1}{2\log(2)}$, $\epsilon\log\mathbb{P}_3\le -R-\frac{1}{2}$.

We have now proved Step 3 and thus Proposition B.4.1. $\square$

Remark B.4.2 *Unlike regular proofs we do not use in Step 2 the Gronwall inequality. Instead we split the norm in many parts and keep the convolution with the group in order to use the (ii) of the Strichartz inequalities.*

Remark B.4.3 *The uniform LDP holds with an extra term $f\left(u^{\epsilon,u_0},\epsilon,t,x\right)$ in the drift. It is needed that there exists (s,ρ) conjugate exponents of an admissible pair $(r(q),q)$ such that for every positive T such that $\|\psi\|_{X^{(T,p)}}<\infty$, $\|f(\psi,\epsilon,.,*)\|_{\mathrm{L}^s(0,T;\mathrm{W}^{1,\rho})}$ is bounded and goes to zero with ϵ. This term may account for damping or amplification going to zero along with the noise.*

B.5 End of the proof of the uniform LDP

We prove hereafter how the almost continuity along with Proposition B.2.2 and B.3.1 allow to prove the uniform LDP.

Suppose that I^{u_0} is the rate function of the LDP, then from Proposition B.3.1, since the sets $(I^{u_0})^{-1}([0,a])$ are the direct image by $\mathbf{S}(u_0,\cdot)$ of the sets C_a which are compact, it is a good rate function.

Now the set A is a Borel set of $\mathcal{E}_\infty$ and u_0 is some initial datum in H^1.

An upper bound. In the case where $\inf_{w\in\overline{A}} I^{u_0}(w)=0$ there is nothing to prove. Otherwise, take $0<a<\inf_{w\in\overline{A}} I^{u_0}(w)$ and $R>a$. Suppose that f is such that $I^W(f)\leq a$, then

$$I^{u_0}(\mathbf{S}(u_0,f))\leq a<\inf_{w\in\overline{A}} I^{u_0}(w),$$

thus $\mathbf{S}(u_0,f)\notin\overline{A}$ and there exists a neighborhood of $\mathbf{S}(u_0,f)$

$$V_{u_0,f}=\{v\in\mathcal{E}_\infty:\ \mathcal{T}(v)>T \text{ and } \|v-\mathbf{S}(u_0,f)\|_{X^{(T,p)}}<\rho_{u_0,f}\}$$

such that $V_{u_0,f}\subset\overline{A}^c$. Also, from Proposition B.4.1, there exists $\epsilon_{u_0,f}$, $\gamma_{u_0,f}$ and $r_{u_0,f}$ positive such that for every $\epsilon\leq\epsilon_{u_0,f}$ and $\tilde{u}_0$ in $B_{\mathrm{H}^1}(u_0,r_{u_0,f})$,

$$\epsilon\log\mathbb{P}\left(\left\|u^{\epsilon,\tilde{u}_0}-\mathbf{S}(u_0,f)\right\|_{X^{(T,p)}}\geq\rho_{u_0,f};\left\|\sqrt{\epsilon}W-f\right\|_{\mathrm{C}\left([0,T];\mathrm{H}^s_{\mathbb{R}}\right)}<\gamma_{u_0,f}\right)\leq -R.$$

Let denote by $O_{u_0,f}$ the set $O_{u_0,f}=B_{\mathrm{C}\left([0,T];\mathrm{H}^s_{\mathbb{R}}\right)}(f,\gamma_{u_0,f})$. The family $(O_{u_0,f})_{f\in C_a}$ is a covering by open sets of the compact set C_a, thus there exists a finite sub-covering of the form $\bigcup_{i=1}^N O_{u_0,f_i}$. We can now write

$$\begin{aligned}\mathbb{P}\left(u^{\epsilon,\tilde{u}_0}\in A\right)\leq\ & \mathbb{P}\left(\left\{u^{\epsilon,\tilde{u}_0}\in A\right\}\cap\left\{\sqrt{\epsilon}W\in\textstyle\bigcup_{i=1}^N O_{u_0,f_i}\right\}\right)\\ & +\mathbb{P}\left(\sqrt{\epsilon}W\notin\textstyle\bigcup_{i=1}^N O_{u_0,f_i}\right)\\ \leq\ & \textstyle\sum_{i=1}^N\mathbb{P}\left(\left\{u^{\epsilon,\tilde{u}_0}\in A\right\}\cap\{\sqrt{\epsilon}W\in O_{u_0,f_i}\}\right)+\mathbb{P}\left(\sqrt{\epsilon}W\notin C_a\right)\\ \leq\ & \textstyle\sum_{i=1}^N\mathbb{P}\left(\left\{u^{\epsilon,\tilde{u}_0}\notin V_{u_0,f_i}\right\}\cap\{\sqrt{\epsilon}W\in O_{u_0,f_i}\}\right)+\exp\left(-\frac{a}{\epsilon}\right),\end{aligned}$$

for $\epsilon\leq\epsilon_0$ for some ϵ_0 positive. Thus, for $\epsilon\leq\epsilon_0\wedge\left(\bigwedge_{i=1}^N\epsilon_{u_0,f_i}\right)$ we obtain for $\tilde{u}_0$ in $B_{\mathrm{H}^1}(u_0,r_{u_0})$ where $r_{u_0}=\bigwedge_{i=1}^N r_{u_0,f_i}$,

$$\mathbb{P}\left(u^{\epsilon,\tilde{u}_0}\in A\right)\leq\ N\exp\left(-\frac{R}{\epsilon}\right)+\exp\left(-\frac{a}{\epsilon}\right),$$

and

$$\epsilon\log\mathbb{P}\left(u^{\epsilon,\tilde{u}_0}\in A\right)\leq\epsilon\log 2+(\epsilon\log N-R)\vee(-a).$$

Finally, there exists r_{u_0} such that for any $\tilde{u}_0$ in $B_{\mathrm{H}^1}(u_0,r_{u_0})$,

$$\overline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(u^{\epsilon,\tilde{u}_0}\in A\right)\leq -a.$$

Since a is arbitrary, we obtain,

$$\overline{\lim}_{\epsilon\to 0,\tilde{u}_0\to u_0}\epsilon\log\mathbb{P}\left(u^{\epsilon,\tilde{u}_0}\in A\right)\le -\inf_{w\in\overline{A}} I^{u_0}(w).$$

A lower bound. Suppose that $\inf_{w\in Int(A)} I^{u_0}(w)<\infty$, otherwise there is nothing to prove, and take w in $Int(A)$ such that $I^{u_0}(w)<\infty$.

The continuity of $\mathbf{S}(u_0,\cdot)$ along with the compactness of the set $C_{I^{u_0}(w)+1}$ give that there exists f such that $w=\mathbf{S}(u_0,f)$ and $I^{u_0}(w)=I^W(f)$. Take $V_{u_0,f}$ an elementary neighborhood of $\mathbf{S}(u_0,f)$ included in A and $O_{u_0,f}$ defined as previously, η positive and $R>I^{u_0}(w)+\eta$. We obtain

$$\begin{aligned}\exp\left(-\tfrac{R-\eta}{\epsilon}\right) &\le \exp\left(-\tfrac{I^W(f)}{\epsilon}\right)\\ &\le \mathbb{P}\left(\sqrt{\epsilon}W\in O_{u_0,f}\right)\\ &\le \mathbb{P}\left(\left\{u^{\epsilon,\tilde{u}_0}\notin V_{u_0,f}\right\}\cap\left\{\sqrt{\epsilon}W\in O_{u_0,f}\right\}\right)+\mathbb{P}\left(u^{\epsilon,\tilde{u}_0}\in A\right).\end{aligned}$$

Thus, there exists r_{u_0} and ϵ_0 positive such that for every $\tilde{u}_0$ in $B_{\mathrm{H}^1}(u_0,r_{u_0})$ and $\epsilon\le\epsilon_0$,

$$-R+\eta\le\epsilon\log 2+\left(\epsilon\log\mathbb{P}\left(u^{\epsilon,\tilde{u}_0}\in A\right)\right)\vee(-R)$$

and there exists $\epsilon_1\le\epsilon_0$ such that for every $\epsilon\le\epsilon_1$,

$$-I^{u_0}(w)\le\epsilon\log 2+\epsilon\log\mathbb{P}\left(u^{\epsilon,\tilde{u}_0}\in A\right).$$

As a consequence, we obtain that for every u_0 in H^1, there exists r_{u_0} positive such that for every $\tilde{u}_0$ in $B_{\mathrm{H}^1}(u_0,r_{u_0})$,

$$\underline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(u^{\epsilon,\tilde{u}_0}\in A\right)\ge -I^{u_0}(w)$$

and

$$\underline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(u^{\epsilon,\tilde{u}_0}\in A\right)\ge -\inf_{w\in Int(A)} I^{u_0}(w)$$

since w in $Int(A)$ is arbitrary.

The uniform LDP follows from the two bounds and Corollary 5.6.15 in [48].

B.6 Applications to the blow-up times

In this section, the equation with a focusing nonlinearity is considered. Then, some solutions of the deterministic equation blow up in finite time for critical or supercritical nonlinearities. If B is a Borel set of $[0,\infty]$,

$\mathbb{P}\left(\mathcal{T}\left(u^{\epsilon,u_0}\right)\in B\right) = \mu^{u^{\epsilon,u_0}}\left(\mathcal{T}^{-1}(B)\right)$. Thus, the uniform LDP gives that for $K \subset\subset \mathrm{H}^1$,

$$-\sup_{u_0\in K}\inf_{u\in Int(\mathcal{T}^{-1}(B))} I^{u_0}(u) \leq \underline{\lim}_{\epsilon\to 0}\epsilon\log\inf_{u_0\in K}\mathbb{P}\left(\mathcal{T}(u^{\epsilon,u_0})\in B\right)$$

and

$$\overline{\lim}_{\epsilon\to 0}\epsilon\log\sup_{u_0\in K}\mathbb{P}\left(\mathcal{T}(u^{\epsilon,u_0})\in B\right) \leq -\inf_{u\in\overline{\mathcal{T}^{-1}(B)},u_0\in K} I^{u_0}(u).$$

Since $\mathcal{T}$ is lower semicontinuous, the sets $(T,\infty]$ and $[0,T]$ are particularly interesting. We recall, see [81] for more details, that for every T positive, $\overline{\mathcal{T}^{-1}((T,\infty])} = \mathcal{E}_\infty$ and $Int\left(\mathcal{T}^{-1}([0,T])\right) = \emptyset$. Thus, for the two types of sets, at least one bound is trivial. Considering approximate blow-up times, see [19], allows us to obtain two interesting bounds and to treat intervals of the form $(S,T]$ where $0\leq S<T$. We do not consider this latter question in this chapter. We finally recall that when $T<\mathcal{T}(u_d^{u_0})$ the LDP gives that $\lim_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(\mathcal{T}(u^{\epsilon,u_0})>T\right)=0$, indeed this is not a large deviation event. We obtain similarly, when $T>\mathcal{T}(u_d^{u_0})$, $\lim_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(\mathcal{T}(u^{\epsilon,u_0})\leq T\right)=0$.

Proposition B.6.1 *If $T<\mathcal{T}_K^i=\inf_{u_0\in K}\mathcal{T}\left(u_d^{u_0}\right)$, where $K\subset\subset \mathrm{H}^1$, then there exists c positive such that*

$$\overline{\lim}_{\epsilon\to 0}\epsilon\log\sup_{u_0\in K}\mathbb{P}\left(\mathcal{T}(u^{\epsilon,u_0})\leq T\right)\leq -c.$$

Proof. Since $\mathcal{T}$ is lower semicontinuous, $\mathcal{T}^{-1}([0,T])$ is a closed set. Suppose that there exists a sequence (u_n,h_n) in $K\times \mathrm{L}^2(0,\infty;\mathrm{L}^2)$ such that $\mathcal{T}\left(\mathbf{S}^c(u_n,h_n)\right)\leq T$ and $\lim_{n\to\infty}h_n=0$. Since K is a compact set we may extract a subsequence $u_{\varphi(n)}$ such that $u_{\varphi(n)}$ converges to some $\tilde{u}$. Also, if we denote by $f_n(\cdot)=\int_0^\cdot \Phi h_n(s)ds$, f_n converges to zero in $\mathrm{C}\left([0,\infty);\mathrm{H}^s_{\mathbb{R}}\right)$ and satisfies $\mathcal{T}\left(\mathbf{S}(u_n,f_n)\right)\leq T$. Moreover, there exists a positive such that for every n in $\mathbb{N}$, f_n in C_a. The semicontinuity of $\mathcal{T}$ along with Proposition B.3.1 give that

$$T\geq \underline{\lim}_{n\to\infty}\mathcal{T}\left(\mathbf{S}(u_{\varphi(n)},f_{\varphi(n)})\right)\geq\mathcal{T}\left(\mathbf{S}(\tilde{u},0)\right)\geq\mathcal{T}_K^i>T,$$

which is contradictory. □

In the following we consider the case $d=2$ or $d=3$ and a cubic nonlinearity, *i.e.* $\sigma=1$. In that case blow-up may occur.

Proposition B.6.2 *Let U^{u_0} be the solution of the free Schrödinger equation with initial datum u_0 in H^s and assume that* $\mathrm{span}\left\{|U^{u_0}(t)|^2, t \in [0,2T]\right\}$ *belongs to the range of Φ for $T > \mathcal{T}\left(u_d^{u_0}\right)$. There exists c positive such that*

$$\underline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(\mathcal{T}(u^{\epsilon,u_0}) > T\right) \geq -c.$$

Remark B.6.3 *It is known that for some Gaussian initial data u_0, see [13], the solutions of NLS blow up in finite time. Also, the solutions of the free equation are smooth and strongly decreasing at infinity, thus we may check that it is possible to define a Hilbert-Schmidt operator Φ satisfying the last assumption.*

Proof. Define F^{u_0} by $F^{u_0}(t) = -\int_0^{t\wedge 2T} |U^{u_0}(s)|^2 ds$. The control is such that $\mathbf{S}(u_0, F^{u_0}) = U^{u_0}$ on $[0,2T]$ thus $\mathcal{T}\left(\mathbf{S}(u_0,F^{u_0})\right) \geq 2T$ since $\mathbf{S}(u_0,F^{u_0})$ does not blow up. Also, F^{u_0} belongs to $\mathrm{C}\left([0,\infty),\mathrm{H}^s_{\mathbb{R}}\right)$ since, for $s > \frac{d}{2}$, H^s is an algebra and U^{u_0} belongs to $\mathrm{C}([0,\infty),\mathrm{H}^s)$. Finally, from the assumption on Φ, there exists h in $\mathrm{L}^2(0,\infty;\mathrm{L}^2)$ setting $h = 0$ after $2T$ such that $\Phi h(s) = |U^{u_0}(s)|^2 \mathbb{1}_{s\leq 2T}$ and F^{u_0} belongs to C_a for some a positive. We thus obtain that F^{u_0} belongs to $\left\{f \in \mathrm{C}\left([0,\infty),\mathrm{H}^s_{\mathbb{R}}\right) : \mathcal{T}\left(\mathbf{S}(u_0,f)\right) > T\right\}$ and that $I^W(F^{u_0}) \leq a < \infty$. □

Remark B.6.4 *A result on compact sets K in H^s for $T > \sup_{u_0\in K}\mathcal{T}\left(u_d^{u_0}\right)$ holds provided that* $\mathrm{span}\left\{|U^{u_0}(t)|^2, t\in[0,2T], u_0 \in K\right\}$ *belongs to the range of Φ restricted to a ball of $\mathrm{L}^2\left(0,2T;\mathrm{L}^2\right)$.*

Appendix C

Small noise asymptotic of the timing jitter in soliton transmission

Abstract: We consider random perturbations of the focusing cubic one dimensional nonlinear Schrödinger equation. The noises, either additive or multiplicative, are white in time and colored in space. In the additive case a "white noise limit" is considered. We study the small noise asymptotic of the tails of the center and mass of a pulse at a fixed coordinate when the initial datum is null or a soliton profile. Our main tools are large deviation results at the level of paths. Upper and lower bounds are obtained from bounds for the optimal control problems derived from the rate function of the large deviation principles. Our results are in perfect agreement with several results from physics. These results had been obtained with arguments which seems difficult to fully justify mathematically. Some results are new.

C.1 Introduction

The nonlinear Schrödinger (NLS) equation occurs as a generic model in many areas of physics and describes the propagation of slowly varying envelopes of a wave packet in media with both nonlinear and dispersive responses. The one-dimensional equation with a cubic focusing nonlinearity is for example a model in the context of long-haul transmission lines in fiber optics; see for example [88] for a derivation of the equation in that context. The variable t stands for the space coordinate and x for some retarded time.

Resulting from a balance between the focusing nonlinearity and the dispersive linear part, localized (here in time) waves propagate, they are called solitons or solitary waves. The functions

$$\sqrt{2}A\text{sech}(A(x-x_0)+2A\Omega t)\exp\left(-i(A^2-\Omega^2)t+i\Omega(x-x_0)+i\theta_0\right) \quad \text{(C.1.1)}$$

where $A > 0$ is the amplitude, Ω is the group velocity or angular carrier frequency, x_0 and θ_0 are respectively the initial position and phase, are solitons. In soliton based amplitude-shifted-keyed systems (ASK) communication systems, solitons are used as information carriers to transmit the datum 0 or 1. A 1 corresponds to the emission of a soliton at time 0 with null velocity $\Psi_A^0(x) = \sqrt{2}A\text{sech}(Ax)$. It is produced by a laser beam. At the far end T of the fiber a receiver records

$$\frac{1}{l}\int_{-\frac{l}{2}}^{\frac{l}{2}} |u^{u_0}(T,x)|^2\,dx,\ u_0 = 0 \text{ or } u_0 = \Psi_A^0,$$

$\left[-\frac{l}{2}, \frac{l}{2}\right]$ is a window in time; l may be chosen small since the wave u^{u_0}, solution of the NLS equation, is localized and remains centered. When the above quantity is above a threshold I_d it is decided that a 1 has been emitted, otherwise it is decided that a 0 has been emitted.

However, it is physically more relevant to consider random perturbations and then error in transmission may occur. Phenomena such as a fluctuating dielectric permittivity, a deviating fiber radius or a random initial shape maybe taken into account in a perturbation term. Moreover noise is somehow intrinsic to such systems.

To counterbalance for loss in the fiber, regularly spaced amplifiers are placed along the line and the distance between amplifiers is small compared to the length of the line. If we suppose that the gain is adjusted to counterbalance exactly for loss, there remains a spontaneous emission noise. This could be justified theoretically thanks to Heisenberg's uncertainty principle. This noise could be modeled as a random external force; see for example [51, 63, 110]. We could formally write the equation as

$$i\frac{\partial u^{\epsilon,u_0}}{\partial t} = \Delta u^{\epsilon,u_0} + |u^{\epsilon,u_0}|^2 u^{\epsilon,u_0} + \sqrt{\epsilon}\xi, \quad \text{(C.1.2)}$$

where ϵ stands for the small noise amplitude, ξ is a complex Gaussian space-time noise and u_0 is the initial datum. The functions are complex valued. Note that this equation also appears in the context of anharmonic atomic chains in the presence of thermal fluctuation; see for example [16].

Other types of amplification among which Raman coupling to thermal phonon, see [52, 59, 98], and four-wave-mixing, see [52, 101], also lead to spontaneaous emission of noise. However in this case the noise enters as a real multiplicative noise. Note that in the case of the Raman amplification a Raman nonlinear response also appears in the equation and the Raman effect also contributes to the Kerr effect, *i.e.* the power law nonlinearity. It is assumed that the extra Raman nonlinear response may be neglected to a first approximation in a treatment of the noise effect on the frequency and thus, by dynamical coupling, on the position of the pulse since it produces essentially a deterministic shift in frequency. The evolution equation may be written formally as

$$i\frac{\partial u^{\epsilon,u_0}}{\partial t} = \Delta u^{\epsilon,u_0} + |u^{\epsilon,u_0}|^2 u^{\epsilon,u_0} + \sqrt{\epsilon} u^{\epsilon,u_0}\xi, \tag{C.1.3}$$

in that case the noise ξ is a real Gaussian noise. Note that this model is also introduced in the context of crystals; see for example [10, 11, 12].

In the presence of noise, the soliton is progressively distorted by the noise, even though it is small, and with small probability an error in transmission may occur in the sense that 1 is discarded. Also, when the noise is additive, it may create from nothing a structure that might be mistaken as a 1.

When a 1 is emitted, it is assumed that two processes are mainly responsible for the loss of the signal: a decrease of the mass

$$\mathbf{N}\left(u^{\epsilon,\Psi_A^0}(T)\right) = \left\|u^{\epsilon,\Psi_A^0}(T)\right\|_{\mathrm{L}^2}^2$$

and a diffusion in position, characterized by the center of the pulse

$$\mathbf{Y}\left(u^{\epsilon,\Psi_A^0}(T)\right) = \int_{\mathbb{R}} x\left|u^{\epsilon,\Psi_A^0}(T,x)\right|^2 dx.$$

The fluctuation of the center results in a shift in the arrival time. It is called timing jitter. The event that for null initial datum a 1 is detected only results from a large fluctuation of the mass.

When the noise is of multiplicative type the mass is invariant and we shall only focus on the timing jitter.

Considering that the probability of sending a 1 is $\frac{1}{2}$, the bit error rate is defined as

$$\frac{1}{2}\mathbb{P}\left(\frac{1}{l}\int_{-\frac{l}{2}}^{\frac{l}{2}}\left|u^{\epsilon,\Psi_A^0}(T,x)\right|^2 dx \le I_d\right) + \frac{1}{2}\mathbb{P}\left(\frac{1}{l}\int_{-\frac{l}{2}}^{\frac{l}{2}}\left|u^{\epsilon,0}(T,x)\right|^2 dx > I_d\right),$$

the probabilities that the measured quantities are below or above the threshold are conditional probabilities. Again, in the case of a multiplicative noise the second conditionnal probability is null. In practical applications, this bit error rate might be less than 10^{-9}. Moreover it is widely admitted that the statistics are not Gaussian. Thus a statistical treatment for inference of the bit error rate requires a theoretical evaluation.

In the physics literature the amplitude of the noise is assumed to be small. Physical techniques often rely on an adiabatic perturbation theory where the pulse is approximated by a soliton ansatz with finite fluctuating collective variables; it requires that the noise is small.

Some articles from physics study the variance of the center; see for example of [16, 59, 88]. In the seminal paper [88] of Gordon and Haus it is obtained that the variance of the center is of the order of T^3 (superdiffusion, *i.e.* stronger than that of the Brownian motion which is linear) and that the fluctuation of the center is connected with a shift in the soliton carrier frequency. It is assumed that the timing jitter is the most troublesome and upper limit of the information rate is derived based on a Gaussian assumption. In [59], the only paper from physics we found on noise induced timing jitter when the noise is multiplicative, a Raman-modified NLS equation is considered; independent complex additive and real multiplicative noises appear both in the equation. The contribution of each noise to the variance of the center is of the order T^3. They however exhibit a different behavior in the initial amplitude A.

Other articles study the deviation from the Gaussian assumption. Again using the perturbation theory of solitons, see for example [89, 94], physicists have obtained that the statistics of the center may be non Gaussian when there is soliton interaction or filtering, see for example [51, 64, 65, 85, 111]. Otherwise it could be considered as Gaussian in the first order only; see for example [2, 51, 97]. In [114] as in [88] the model is a juxtaposition of deterministic evolutions with randomly perturbed initial initial data in between amplifiers. The log of the tails of the amplitude and center are evaluated numerically via an importance sampled Monte Carlo estimator. Simulations are obtained from a distribution where the small probability event is a central event; they are weighted by a likelihood ratio weight. It is obtained that the log of tails of the amplitude only differs significantly from that of Gaussian tails. Note that we may expect to use the numerical methodology based on a genealogical particle analysis developed in [47]. In this reference the importance sampling and Monte Carlo methodologies are compared to a particle system approach and it is applied to the estimation of probability of rare events due to polarization-mode dispersion in optical fibers.

In [51, 63, 110], probability density functions (PDF) are examinated. In [63] the PDF of the joint law of the mass and center at coordinate T, when the initial datum is a soliton profile, are approximated from a PDF of the random parameters of a solution described as a soliton with a finite set of fluctuating parameters. The parameters are assumed to evolve according to dynamically coupled SDEs. This latter PDF is obtained via a saddle point approximation of a corresponding finite dimensional Martin-Siggia-Rose effective action. The complete infinite dimensional effective action, see for example [95] is not treated. The PDF of the amplitude (a multiple of the mass with the parametrization) is obtained when the initial datum is null. The probability of loosing a 1 is numerically evaluated under the assumption of a very large window. In [51] the Fokker-Planck equation is used to obtain the PDF of the mass at T. In [110] a similar result is obtained. However the PDF of the marginal law of the center has not been evaluated.

Note that infinite dimensional effective actions in physics are intimately related to the rate function of a sample path large deviation principle (LDP). Paths minimizing the action for certain configurations of the system are called optimal fluctuations or instantons, see also for example [9, 129]. Note that in [67], where the large deviations approach is adopted, the problem of transitions between stable equilibrium configurations (tunnelling) of unforced nonlinear heat equations in the limit of small noise is studied. The most likely transitions are the instantons from quantum mechanics; they are saddle points of the equilibrium action functional related to the rate function of the sample path LDP. Exit from neighborhoods of zero for weakly damped stochastic NLS equations is studied in the article [84].

In the present article we apply sample path LDPs to the study of the tails of the law of the mass and center of the pulse at the end of the fiber. We thus study cumulative distribution functions (CDFs) instead of PDFs but do not study the bulk of the distribution. As we will see, we are not able to treat mathematically the case of the space-time white noise which is mainly used in the physical models. We thus restrict ourselves to noises that are colored in space. In the case of a noise of additive type we will consider sequences of noises that mimic the white noise in the limit. The log of the tails in the limit of small noise are of the order of the opposite of the infima of a functional derived from the rate functions of the LDPs divided by the noise amplitude. The infima are optimal control problems. We give upper and lower bounds using energy inequalities and modulated solitons. The two bounds mostly differ up to multiplicative constants and the orders in T and A are compared to that of the physicists.

C.2 Notations and preliminaries

For $p \geq 1$, L^p is the classical Lebesgue space of complex valued functions on $\mathbb{R}$ and $\mathrm{W}^{1,p}$ is the associated Sobolev space of L^p functions with first order derivatives, in the sense of distributions, in L^p. If I is an interval of $\mathbb{R}$, $(E, \|\cdot\|_E)$ a Banach space and r belongs to $[1,\infty]$, then $\mathrm{L}^r(I;E)$ is the space of strongly Lebesgue measurable functions f from I into E such that $t \to \|f(t)\|_E$ is in $\mathrm{L}^r(I)$. The space L^2 with the inner product defined by $(u,v)_{\mathrm{L}^2} = \mathfrak{Re} \int_{\mathbb{R}} u(x)\overline{v}(x)dx$ is a Hilbert space. The Sobolev spaces H^s are the Hilbert spaces of functions of L^2 with partial derivatives up to order s in L^2. When s is fractional it is defined classically via the Fourier transform. When the functions are real valued we specify it, for example we write $\mathrm{H}^s(\mathbb{R},\mathbb{R})$. The following Hilbert spaces of spatially localized functions

$$\Sigma = \left\{ f \in \mathrm{H}^1 : \ x \mapsto x f(x) \in \mathrm{L}^2 \right\},$$

$$\Sigma^{\frac{1}{2}} = \left\{ f \in \mathrm{H}^1 : \ x \mapsto \sqrt{|x|} f(x) \in \mathrm{L}^2 \right\}$$

are also introduced and endowed with the norms

$$\|f\|_{\Sigma}^2 = \|f\|_{\mathrm{H}^1}^2 + \|x \mapsto x f(x)\|_{\mathrm{L}^2}^2 ,$$

$$\|f\|_{\Sigma^{\frac{1}{2}}}^2 = \|f\|_{\mathrm{H}^1}^2 + \left\| x \mapsto \sqrt{|x|} f(x) \right\|_{\mathrm{L}^2}^2 .$$

We denote by $\|\Phi\|_{\mathcal{L}_c(A,B)}$ the norm of Φ as a linear continuous operator from A to B, where A and B are normed vector spaces. We recall that Φ is a Hilbert-Schmidt operator from H to $\tilde{H}$, where H and $\tilde{H}$ are Hilbert spaces, if it is a linear continuous operator such that, given a complete orthonormal system $(e_j^H)_{j=1}^{\infty}$ of H, $\sum_{j=1}^{\infty} \|\Phi e_j^H\|_{\tilde{H}}^2 < \infty$. We will denote by $\mathcal{L}_2(H,\tilde{H})$ the space of Hilbert-Schmidt operators from H to $\tilde{H}$ endowed with the norm

$$\|\Phi\|_{\mathcal{L}_2(H,\tilde{H})} = \mathrm{tr}\,(\Phi\Phi^*) = \sum_{j=1}^{\infty} \|\Phi e_j^H\|_{\tilde{H}}^2 .$$

We also recall that a cylindrical Wiener process W_c in a Hilbert space H is such that for any complete orthonormal system $(e_j)_{j=1}^{\infty}$ of H, there exists a sequence of independent Brownian motions $(\beta_j)_{j=1}^{\infty}$ such that $W_c = \sum_{j=1}^{\infty} \beta_j e_j$. This sum does not converge in H but in any Hilbert space U such that the embedding $H \subset U$ is Hilbert-Schmidt. The image of the process W_c by a linear mapping Φ on H is a well defined process in H when the mapping is Hilbert-Schmidt on H, *i.e.* $\Phi \in \mathcal{L}_2(H) = \mathcal{L}_2(H,H)$. Then,

$W = \Phi W_c$ is such that $W(1)$ is well defined with a covariance operator $\Phi\Phi^*$.

We recall that a rate function I is a lower semicontinuous function and that a good rate function I is a rate function such that for every positive c, $\{x: \ I(x) \leq c\}$ is a compact set.

Let us now recall some mathematical aspects of the stochastic NLS equations. The equations, written as SPDEs in the Itô form, are in the additive case

$$i\mathrm{d}u^{\epsilon,u_0} - (\Delta u^{\epsilon,u_0} + |u^{\epsilon,u_0}|^2 u^{\epsilon,u_0})\mathrm{d}t = \sqrt{\epsilon}\mathrm{d}W, \tag{C.2.1}$$

and in the multiplicative case

$$i\mathrm{d}u^{\epsilon,u_0} - (\Delta u^{\epsilon,u_0} + |u^{\epsilon,u_0}|^2 u^{\epsilon,u_0})\mathrm{d}t = \sqrt{\epsilon}u^{\epsilon,u_0} \circ \mathrm{d}W. \tag{C.2.2}$$

The symbol $\circ$ stands for the Stratonovich product. In the case of equation (C.2.2), see [37], the mass

$$\mathbf{N}\left(u^{\epsilon,u_0}(t)\right) = \|u^{\epsilon,u_0}(t)\|^2_{\mathrm{L}^2}, \quad t > 0$$

is a conserved quantity. Precise assumptions on Φ such that $W = \Phi W_c$ are made below. These equations are supplemented with an initial datum

$$u^{\epsilon,u_0}(0) = u_0.$$

In this paper, we consider initial data in $\Sigma \subset \mathrm{H}^1$ and work with the solution constructed in [37]. Since we work with a subcritical non linearity, we could also consider solutions in L^2 with initial data in L^2. However, the H^1-setting is preferred in order to be able to consider the spaces Σ and $\Sigma^{\frac{1}{2}}$ and study the center of the pulse

$$\mathbf{Y}(u^{\epsilon,u_0}(t)) = \int_{\mathbb{R}} x|u^{\epsilon,u_0}(t,x)|^2 dx, \quad t > 0,$$

defined when $u^{\epsilon,u_0}(t)$ belongs to $\Sigma^{\frac{1}{2}}$.

We are concerned by weak solutions or equivalently by mild solutions which, in the additive case, satisfy

$$\begin{aligned} u^{\epsilon,u_0}(t) = \quad & U(t)u_0 - i\int_0^t U(t-s)(|u^{\epsilon,u_0}(s)|^2 u^{\epsilon,u_0}(s))ds \\ & -i\sqrt{\epsilon}\int_0^t U(t-s)dW(s) \end{aligned} \tag{C.2.3}$$

where $(U(t))_{t\in\mathbb{R}}$ stands for the Schrödinger group, $U(t)=e^{-it\Delta},\ t\in\mathbb{R}$. The last term is called the stochastic convolution. In the multiplicative case, the mild equation is

$$\begin{aligned} u^{\epsilon,u_0}(t)= \quad & U(t)u_0 - i\int_0^t U(t-s)(|u^{\epsilon,u_0}(s)|^2 u^{\epsilon,u_0}(s))ds \\ & -i\sqrt{\epsilon}\int_0^t U(t-s)u^{\epsilon,u_0}(s)dW(s) - \frac{i\epsilon}{2}\int_0^t U(t-s)F_\Phi u^{\epsilon,u_0}(s)ds \end{aligned} \tag{C.2.4}$$

where the stochastic integral is a Itô integral and, given $(e_j)_{j=1}^\infty$ an orthonormal basis of L^2, $F_\Phi(x)=\sum_{j=1}^\infty(\Phi e_j)^2(x)$. The term $\frac{\epsilon}{2}F_\Phi(x)$ is the Itô correction.

The noise is the time derivative in the sense of distributions of the Wiener process W. It corresponds to a white noise in time. A space-time white noise would correspond to Φ equal to the identity. We cannot handle such rough noises and make the assumption that the two noises are colored in space. The basic limitation is that, unlike semi-groups like the Heat semi-group, the Schrödinger group is an isometry and does not allow smoothing in the Sobolev spaces based on L^2. For instance, in the additive case, it can be seen that the stochastic convolution is a well defined process with paths in L^2 if and only if Φ is a Hilbert-Schmidt operator on L^2.

In fact, we make even stronger assumptions. In the additive case we assume that W is a Wiener process on Σ. In the multiplicative case, it is imposed that W is a Wiener process on $\mathrm{H}^s(\mathbb{R},\mathbb{R})$ where s satisfies $s>\frac{3}{2}$.

We know that the Cauchy problem is globally well posed in H^1; see [37] for a general discussion on the local well posedness and the global existence for more general nonlinearities and dimensions. Note that the present deterministic NLS equation is integrable thanks to the inverse scattering method. We will not use these techniques in the article. Results on the influence of the noise on the blow-up time, for more general nonlinearities and dimensions are given in [39, 40]. In [13, 43] the ideal white noise and results on the influence of a noise on the blow-up are studied numerically.

Sample path LDPs for stochastic NLS equations are proved in [81, 82]. These LDPs do not allow to treat the center of the solution and we shall consider LDPs in $\mathrm{C}\left([0,T];\Sigma^{\frac{1}{2}}\right)$ where T is positive (the length of the fiber line). The rate function of the LDP in the additive case is defined in terms of the mild solution of the control problem

$$\left\{ \begin{array}{l} i\frac{\mathrm{d}u}{\mathrm{d}t}=\Delta u+|u|^2u+\Phi h, \\ u(0)=u_0\ \in\Sigma \text{ and } h\in\mathrm{L}^2\left(0,T;\mathrm{L}^2\right). \end{array}\right. \tag{C.2.5}$$

We denote the solution by $u = \mathbf{S}^{a,u_0}(h)$. The mapping $h \to \mathbf{S}^{a,u_0}(h)$ is called the skeleton and (C.2.5) the skeleton equation.

In the multiplicative case, the controlled equation is

$$i\frac{\mathrm{d}u}{\mathrm{d}t} = \Delta u + |u|^2 u + u\Phi h, \tag{C.2.6}$$

whose mild solution is denoted by $u = \mathbf{S}^{m,u_0}(h)$. The mapping $\mathbf{S}^{m,u_0}$ is again called the skeleton and (C.2.6) the skeleton equation.

In this article, when describing properties which hold both in the additive and multiplicative cases, we use the symbol $\mathbf{S}(u_0, h)$ to denote either $\mathbf{S}^{a,u_0}(h)$ or $\mathbf{S}^{m,u_0}(h)$.

Let us now state the sample path LDPs. The proof is given in the annex.

Theorem C.2.1 *Assume that Φ belongs to $\mathcal{L}_2(\mathrm{L}^2, \Sigma)$ in the additive case and $\Phi \in \mathcal{L}_2(\mathrm{L}^2, \mathrm{H}^s(\mathbb{R}, \mathbb{R}))$ with $s > 3/2$ in the multiplicative case. Assume also that the initial datum u_0 is in Σ. Then the solutions of the stochastic nonlinear Schrödinger equations* (C.2.3) *and* (C.2.4) *are almost surely in* $\mathrm{C}([0,T]; \Sigma^{\frac{1}{2}})$*. Moreover, they define* $\mathrm{C}([0,T]; \Sigma^{\frac{1}{2}})$ *random variables and their laws $\left(\mu^{u^{\epsilon,u_0}}\right)_{\epsilon>0}$ satisfy a LDP of speed ϵ and good rate function*

$$I^{u_0}(w) = \frac{1}{2} \inf_{h \in \mathrm{L}^2(0,T;\mathrm{L}^2):\ w = \mathbf{S}(u_0,h)} \|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)},$$

where $\mathbf{S}(u_0, \cdot) = \mathbf{S}^{a,u_0}(\cdot)$ in the additive case and $\mathbf{S}(u_0, \cdot) = \mathbf{S}^{m,u_0}(\cdot)$ in the multiplicative case, and with the convention that $\inf \emptyset = \infty$*. It means that for every Borel set B of* $\mathrm{C}\left([0,T]; \Sigma^{\frac{1}{2}}\right)$*, we have the lower bound*

$$-\inf_{w \in Int(B)} I^{u_0}(w) \le \underline{\lim}_{\epsilon \to 0} \epsilon \log \mathbb{P}\left(u^{\epsilon,u_0} \in B\right)$$

and the upper bound

$$\overline{\lim}_{\epsilon \to 0} \epsilon \log \mathbb{P}\left(u^{\epsilon,u_0} \in B\right) \le -\inf_{w \in \overline{B}} I^{u_0}(w).$$

These sample path LDPs allow for example to evaluate the probability that, originated from a soliton profile

$$\Psi^0_A(x) = \sqrt{2} A \operatorname{sech}(Ax),$$

the random solution be significantly different from the deterministic soliton solution

$$\Psi_A(t,x) = \Psi^0_A(x) \exp\left(-iA^2 t\right).$$

Indeed, for T, δ and η positive and ϵ small enough, the LDP implies that

$$\exp\left(-\frac{C_1}{\epsilon}\right) \leq \mathbb{P}\left(\left\|u^{\epsilon,\Psi_A^0} - \Psi_A\right\|_{C([0,T];\Sigma^{\frac{1}{2}})} > \delta\right) \leq \exp\left(-\frac{C_2}{\epsilon}\right),$$

where

$$C_1 = \inf_{w:\ \|w-\Psi_A\|_{C([0,T];\Sigma^{\frac{1}{2}})} > \delta} I^{\Psi_A^0}(w) + \eta$$

and

$$C_2 = \inf_{w:\ \|w-\Psi_A\|_{C([0,T];\Sigma^{\frac{1}{2}})} \geq \delta} I^{\Psi_A^0}(w) - \eta.$$

Recall that, since the rate function is a good rate function, if B is a closed set and $\inf_{w\in B} I^{\Psi_A^0}(w) < \infty$, then there is an f in B, optimal fluctuation, such that $I^{\Psi_A^0}(f) = \inf_{w\in B} I^{\Psi_A^0}(w)$. Thus if B does not contain the deterministic solution then necessarily $\inf_{w\in B} I^{\Psi_A^0}(w) > 0$. Consequently η may be chosen such that C_2 is positive and the above probability of a deviation from the deterministic path is exponentially small in the small ϵ limit.

In this article we are interested in estimating the probability of particular deviations from the deterministic paths. Namely, we wish to study how the mass and the center of a solution at coordinate T deviate from their value in the "frozen" deterministic system (*i.e.* when $\epsilon = 0$). In the absence of noise, the mass is a conserved quantity and for initial data being either 0 or Ψ_A^0 the center remains equal to zero.

We know from [81] that we may push forward by continuity the LDP for the paths to a LDP for the mass at T and obtain a LDP with speed ϵ and good rate function for an initial datum u_0

$$I_N^{u_0}(m) = \frac{1}{2}\inf_{h\in L^2(0,T;L^2):\ \mathbf{N}(\mathbf{S}^{a,u_0}(h)(T))=m}\left\{\|h\|^2_{L^2(0,T;L^2)}\right\}.$$

In the case of a multiplicative noise, the mass is a conserved quantity. Thus, in this case, the mass cannot deviate from the deterministic behavior.

Similarly, the mapping $\mathbf{Y}$ is continuous from $\Sigma^{\frac{1}{2}}$ into $\mathbb{R}$. We may thus define by direct image the measures $\left(\mu^{\mathbf{Y}(u^{\epsilon,u_0}(T))}\right)_{\epsilon>0}$ for an initial datum u_0 in Σ. We obtain by contraction that they satisfy a LDP of speed ϵ and good rate function

$$I_Y^{u_0}(y) = \frac{1}{2}\inf_{h\in L^2(0,T;L^2):\ \mathbf{Y}(\mathbf{S}(u_0,h)(T))=y}\left\{\|h\|^2_{L^2(0,T;L^2)}\right\},$$

the skeleton $\mathbf{S}$ is either that of the additive or multiplicative case.

Let us briefly explain our strategy to estimate the probability of some event. Let us consider for instance the event $D_\epsilon = \left\{\mathbf{Y}\left(u^{\epsilon,0}(T)\right) \in [a,b]\right\}$ where $[a,b]$ is an interval which does not contain 0. We use the LDP to obtain

$$-\inf_{y\in(a,b)} I_Y^0(y) \leq \underline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(D_\epsilon\right) \leq \overline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(D_\epsilon\right) \leq -\inf_{y\in[a,b]} I_Y^0(y). \tag{C.2.7}$$

To estimate the upper bound, we use energy type inequalities. These give estimates of the minimum L^2 norm of the control h required to change the deterministic behavior and have the center in $[a,b]$ at time T. Namely, we obtain a constant c such that

$$\text{if } \mathbf{Y}\left(S(u_0,h)(T)\right) \in [a,b] \text{ then } \frac{1}{2}\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \geq c.$$

This clearly implies

$$\overline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(D_\epsilon\right) \leq -c.$$

The second step is to find a particular function h such that $\mathbf{Y}(S(u_0,h)(T)) \in (a,b)$ and $\tilde{c} = \frac{1}{2}\|h_J\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)}$ is as small as possible. Then

$$-\tilde{c} \leq \underline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(D_\epsilon\right).$$

In this second step, we are led to solve a control problem.

The difficulty is to have sufficiently sharp energy estimates and to find a good solution to the control problem so that c and $\tilde{c}$ are as close as possible. We see below that we are able to do so in some interesting situations and derive good estimates on such probabilities.

Note also that proceeding as in [81] for the mass, we may prove in the additive case that $\inf_{y\in J} I_Y^{u_0}(y) < \infty$ for every nonempty interval J and any u_0 provided the range of Φ is dense. Indeed, for every real number a, a solution of the form $u(t,x) = (1+atx)u_0$ satisfies $\mathbf{Y}\left(u(T)\right) = \frac{aT\pi^2}{3}$. Plugging this solution into equation (C.2.5), we find a control such that the solution reaches any interval at time T. Using the continuity of $h \mapsto \mathbf{Y}\left(\mathbf{S}^{a,u_0}(h)(T)\right)$ from $\mathrm{L}^2(0,T;\mathrm{L}^2)$ into $\mathbb{R}$ and the density of the range of Φ, we obtain $\inf_{y\in J} I_Y^{u_0}(y) < \infty$. This shows that in this case the two extreme bounds in (C.2.7) are finite implying that $\mathbb{P}(D_\epsilon)$ goes to zero exponentially fast when ϵ goes to 0.

Remark C.2.2 *Also, using similar arguments as in [81], we can prove that for every positive R besides an at most countable set of points, we can replace* $\underline{\lim}$ *and* $\overline{\lim}$ *by* lim *in the LDP and obtain*

$$\begin{aligned}&\lim_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(\mathbf{Y}\left(u^{\epsilon,u_0}(T)\right) \geq R\right)\\ &= -\tfrac{1}{2\epsilon}\inf_{h\in \mathrm{L}^2(0,T;\mathrm{L}^2):\ \mathbf{Y}(\mathbf{S}(u_0,h)(T))\geq R}\left\{\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)}\right\}\end{aligned}$$

$$\begin{aligned}&\lim_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(\mathbf{Y}\left(u^{\epsilon,u_0}(T)\right)\le -R\right)\\&=-\tfrac{1}{2\epsilon}\inf_{h\in \mathrm{L}^2(0,T;\mathrm{L}^2):\ \mathbf{Y}(\mathbf{S}(u_0,h)(T))\le -R}\left\{\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)}\right\}.\end{aligned}$$

This uses the fact that a monotone and bounded function is continuous almost everywhere.

We end this section with some remarks which will be useful in the development of our method when we consider the center of the solution. Let us consider an initial datum is Ψ_A^0. The probability of tail events of the center are related to the behavior of $\mathbf{Y}\left(S(\Psi_A^0,h)\right)$. If $h\neq 0$, $S(\Psi_A^0,h)(t)\neq\Psi_A$ and the center may move. An equation for the motion of the center is given in [141] in the case of an external potential. The first step consists in multiplying the controlled PDE by $-ix\overline{u}$, taking the real part, and integrating by part the term involving the Laplace operator. We then obtain for the controlled PDE associated to the multiplicative case

$$\left[\mathbf{Y}\left(\mathbf{S}^{m,\Psi_A^0}(h)(t)\right)\right]'=2\mathfrak{Re}\left(i\int_{\mathbb{R}}\overline{\mathbf{S}^{m,\Psi_A^0}(h)(t,x)}\partial_x\mathbf{S}^{m,\Psi_A^0}(h)(t,x)dx\right),\tag{C.2.8}$$

while in the additive case we obtain

$$\begin{aligned}\left[\mathbf{Y}\left(\mathbf{S}^{a,\Psi_A^0}(h)(t)\right)\right]'=&\quad 2\mathfrak{Re}\left(i\int_{\mathbb{R}}\overline{\mathbf{S}^{a,\Psi_A^0}(h)(t,x)}\partial_x\mathbf{S}^{a,\Psi_A^0}(h)(t,x)dx\right)\\&-2\mathfrak{Re}\left(i\int_{\mathbb{R}}x\overline{\mathbf{S}^{a,\Psi_A^0}(h)(t,x)}\left(\Phi h\right)(t,x)dx\right).\end{aligned}\tag{C.2.9}$$

The quantity

$$\mathbf{P}(u)=2\mathfrak{Re}\left(i\int_{\mathbb{R}}\overline{u}(x)\partial_x u(x)dx\right),\ u\in\mathrm{H}^1.$$

on the right hand side of (C.2.8) and (C.2.9) is usually called the momentum.

As a consequence of (C.2.8) we see that in the multiplicative case, the center of the solution of the control problem cannot move unless its phase depends on the space variable. For instance, if the control is chosen so that the solution $\mathbf{S}^{a,\Psi_A^0}(h)(t)$ is a modulated soliton of type (C.1.1) with varying amplitude and group velocity,

$$\begin{aligned}\mathbf{S}^{a,\Psi_A^0}(h)(t)=&\quad\sqrt{2}A(t)\mathrm{sech}(A(t)(x-x_0)+2A(t)\Omega(t)t)\\&\exp\left(-i(A(t)^2-\Omega(t)^2)t+i\Omega(t)(x-x_0)+i\theta_0\right)\end{aligned}$$

we have the well known identity

$$\left[\mathbf{Y}\left(\mathbf{S}^{m,\Psi_A^0}(h)(t)\right)\right]'=-2\Omega(t)\mathbf{N}(\mathbf{S}^{m,\Psi_A^0}(h)(t))=-8\Omega(t)A(t).$$

It will be convenient to choose controlled solutions of the form above. Since the initial datum is Ψ_A^0, we necessarily have $\Omega(0)=0$, hence Ω cannot be chosen constant. We will see that it is sufficient to have a constant amplitude A in order to get sharp bounds. Thus we will use modulated solitons as solutions of the controlled problem with constant amplitude when studying the motion of the center.

The first idea to find a control giving a solution whose center or mass verify some desired property is to take the above modulated soliton and plug it into the skeleton equation. This gives an explicit form of the control in terms of the various parameters. Then, we compute the space-time L^2 norm of this control. We obtain a function of the parameters which we can try to minimize thanks to the calculus of variations. This approach is not easy to perform, the function to minimize has a complicated form and is often singular. Thus, we also have chosen a simpler approach which consists in finding directly controls giving solutions with the desired properties. Note however that the calculus of variations approach has allowed us to guess the form of the modulated soliton we should choose.

Let us consider the following controlled nonlinear Schrödinger equation

$$i\frac{\mathrm{d}u}{\mathrm{d}t}=\Delta u+|u|^2u+\lambda(t)xu \tag{C.2.10}$$

with initial datum Ψ_A^0. The function λ is taken in $\mathrm{L}^1(0,T;\mathbb{R})$. This corresponds to the multiplicative skeleton equation with $\Phi h=\lambda(t)x$ or to the additive one with $\Phi h=\lambda(t)xu$. We use well known transformation to compute explicitly the solution of (C.2.10) which we denote by $\Psi_{A,\lambda}$. We first may check that the functions v_1 and v_2 defined by $v_1(t,x)=\exp\left(i\left(\int_0^t\lambda(s)ds\right)x\right)u(t,x)$ and $v_2(t,x)=\exp\left(-i\int_0^t\left(\int_0^s\lambda(\tau)d\tau\right)^2ds\right)v_1(t,x)$ (gauge transform) satisfy the PDEs

$$i\frac{\partial v_1}{\partial t}=\frac{\partial^2 v_1}{\partial x^2}+|v_1|^2v_1-\left(\int_0^t\lambda(s)ds\right)^2v_1-2i\left(\int_0^t\lambda(s)ds\right)\frac{\partial v_1}{\partial x}$$

and

$$i\left(\frac{\partial v_2}{\partial t}+2\left(\int_0^t\lambda(s)ds\right)\frac{\partial v_2}{\partial x}\right)=\frac{\partial^2 v_2}{\partial x^2}+|v_2|^2v_2$$

with initial datum Ψ_A^0. We conclude using the methods of characteristics that v_3 defined by

$$v_3(t,x)=v_2\left(t,x+2\int_0^t\int_0^s\lambda(u)duds\right)$$

is a solution of the usual NLS equation with initial datum Ψ_A^0. Thus we obtain that $v_3(t,x)=\Psi_A(t,x)$ and that the solution of the Cauchy problem associated to (C.2.10) is

$$\Psi_{A,\lambda}(t,x)=\sqrt{2}A\mathrm{sech}\left(A\left(x-2\int_0^t\int_0^s\lambda(\tau)d\tau ds\right)\right)$$
$$\exp\left[-iA^2t+i\int_0^t\left(\int_0^s\lambda(\tau)d\tau\right)^2ds-ix\int_0^t\lambda(s)ds+2i\left(\int_0^t\lambda(s)ds\right)\left(\int_0^t\int_0^s\lambda(\tau)d\tau ds\right)\right].$$

We obtain a modulated soliton with group velocity given by $\Omega(t)=\int_0^t\lambda(s)ds$. In the additive case, it is possible to obtain a control such that the solution has same center and group velocity and such that the space-time L^2 norm of the control is simpler to compute. It is obtained thanks to the observation that using the gauge transform the solution of the Cauchy problem

$$\begin{cases} i\frac{\mathrm{d}v}{\mathrm{d}t}=\Delta v+|v|^2v+\lambda(t)\left(x-2\int_0^t\int_0^s\lambda(\tau)d\tau ds\right)v \\ v(0)=\Psi_A^0, \end{cases} \tag{C.2.11}$$

is given by

$$\tilde{\Psi}_{A,\lambda}(t,x)=\exp\left(2i\int_0^t\lambda(s)\int_0^s\int_0^\tau\lambda(\sigma)d\sigma d\tau ds\right)\Psi_{A,\lambda}(t,x).$$

Remark C.2.3 *Note that, for the controls chosen above, relation* (C.2.8) *holds also in the additive case. Thus the second term in* (C.2.9) *which, at first glance, could be useful to act on the center is in fact useless.*

Also, it could be thought that the choice of more complicated group velocities could be useful. We have tried to consider a space dependent group velocity but the calculus of variations approach shows that optimality is reached when it does not depend on space.

C.3 Tails of the the mass and center with additive noise

In the case of an additive noise, both the mass and center may deviate from the deterministic behavior and result in error in transmission.

We shall study tails and thus the probability of a deviation from the mean. The constant R will quantify this deviation. We are not really interested in large R. In practice R may be assumed to be in $(0,4)$. But, since ϵ goes to zero and the factor in the exponential should be multiplied by $\frac{1}{\epsilon}$ while R is of order 1. It results in very unlikely events. These significant excursions of the mass and position are exactly large deviation events.

Moreover another parameter is particularly interesting. It is T the length of the fiber optical line. It is assumed to be large. For example we could think of a fiber optical line between Europe and America.

We first recall the results obtained in [81] for the tails of mass of the pulse at the end of the line. The initial datum may be $u_0 = 0$ or $u_0 = \Psi$ where $\Psi(x) = \sqrt{2}\text{sech}(x)$. We could consider a soliton profile with any amplitude A as well but for simplicity, we consider the case $A = 1$. However we consider the parameter A for the timing jitter in order to compare with results from physics.

Let us begin with upper bounds of the tails. As already mentioned, they are obtained thanks to energy estimates. For the second bound we consider the case of the emission of a signal. In that case only a decrease of the mass is troublesome and causes in error in transmission. Thus the bound given only accounts for a significant decrease of the mass.

Proposition C.3.1 *For every positive T and R (R in $(0,4)$ for the second inequality) and every operator Φ in $\mathcal{L}_2(\mathrm{L}^2, \mathrm{H}^1)$, the following inequalities hold*

$$\overline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(\mathbf{N}\left(u^{\epsilon,0}(T)\right) \geq R\right) \leq -\frac{R}{8T\|\Phi\|^2_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{L}^2)}},$$

$$\overline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(\mathbf{N}\left(u^{\epsilon,\Psi}(T)\right) - 4 < -R\right) \leq -\frac{R^2}{8T\|\Phi\|^2_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{L}^2)}(4+R)}.$$

Proof. We only give a sketch of the proof. Details can be found in [81]. We treat the first inequality. The proof for the second inequality is similar. Multiplying by $-i\overline{u}$ the equation

$$i\frac{\mathrm{d}u}{\mathrm{d}t} - \Delta u - \lambda|u|^2 u = \Phi h,$$

integrating over time and space and taking the real part gives, for $t \in [0,T]$,

$$\|\mathbf{S}^{a,0}(h)(t)\|^2_{\mathrm{L}^2} - \|u_0\|^2_{\mathrm{L}^2} = 2\mathfrak{Re}\left(-i\int_0^t \int_{\mathbb{R}} \left((\Phi h)(s,x)\overline{\mathbf{S}^{a,0}(h)(s,x)}\right) dxds\right). \tag{C.3.1}$$

We first integrate once more with respect to $t \in [0,T]$ and use the Cauchy-Schwarz inequality to obtain

$$\left(\int_0^T \|\mathbf{S}^{a,0}(h)(s)\|^2_{\mathrm{L}^2} ds\right)^{1/2} \leq 2T\|\Phi\|_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{L}^2)} \left(\int_0^T \|h(s)\|^2_{\mathrm{L}^2} ds\right)^{1/2}.$$

Then, taking $t = T$ in (C.3.1), using again the Cauchy-Schwarz inequality and the above bound, we deduce

$$\|\mathbf{S}^{a,0}(h)(T)\|_{\mathrm{L}^2}^2 \le 4T\|\Phi\|_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{L}^2)}^2 \int_0^T \|h(s)\|_{\mathrm{L}^2}^2 ds.$$

It follows

$$\begin{aligned} I_N^0(m) &= \frac{1}{2} \inf_{h\in \mathrm{L}^2(0,T;\mathrm{L}^2):\ \mathbf{N}(\mathbf{S}^{a,0}(h)(T))=m} \left\{\|h\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}^2\right\} \\ &\ge \frac{x}{8T\|\Phi\|_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{L}^2)}^2}. \end{aligned}$$

Now, by the LDP on the mass, we have

$$\overline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(\mathbf{N}\left(u^{\epsilon,0}(T)\right) \ge R\right) \le - \inf_{x\in[R,\infty]} I_N^{u_0}(m)$$

and the result follows. □

Let us now consider lower bounds. We use modulated solitons as solutions of the controlled equation. We have found that it is sufficient that only the amplitude is varying. We take the solution of (C.2.5) of the form

$$\sqrt{2}A(t)\exp\left(-i\int_0^t A^2(s)ds\right)\mathrm{sech}(A(t)x). \tag{C.3.2}$$

The singular Euler-Lagrange equation given by the calculus of variations when minimizing the energy of the controls giving such solutions has allowed to guess a good parametrization when the initial datum is either 0 or Ψ. Define the following sets of time dependent functions, depending on a set of parameters D,

$$\mathcal{A}_D^1 = \left\{A\,:\,[0,T]\to\mathbb{R},\ \text{there exists } \tilde{R}\in D \text{ such that } A(t) = \tilde{R}\left(\frac{t}{2T}\right)^2\right\}$$

and

$$\begin{aligned}\mathcal{A}_D^2 = \Bigg\{ & A\,:\,[0,T]\to\mathbb{R},\ \text{there exists } \tilde{R}\in D \text{ such that} \\ & A(t) = \left(8-\tilde{R}-4\sqrt{4-\tilde{R}}\right)\left(\frac{t}{2T}\right)^2 + \left(-4+2\sqrt{4-\tilde{R}}\right)\frac{t}{2T}+1\Bigg\}.\end{aligned}$$

Modulated amplitude taken in $\mathcal{A}_D^1$ or $\mathcal{A}_D^2$ set are associated to controls in the set

$$\mathcal{C}_D^i = \Big\{ h \in \mathrm{L}^2(0,T;\mathrm{L}^2), \text{ there exists } A \in \mathcal{A}_D^i$$
$$h(t,x) = i\frac{A'(t)}{A(t)}\Psi_A(t,x) - i\sqrt{2}A'(t)\exp\left(-i\int_0^t A^2(s)ds\right)A(t)x\frac{\sinh}{\cosh^2}\left(A(t)x\right)\Big\}$$

where $i=1$ or $i=2$.

We have the following proposition whose proof follows from the lower bound of the LDP for the mass. The proof is given in [81]. It uses that the infimum of the rate function is smaller than the infimum on the smaller sets of controls $\mathcal{C}_D^1$ and $\mathcal{C}_D^2$ corresponding to well-chosen modulated amplitudes. The assumptions can easily be fulfilled. They are made to be as close as possible to the space-time white noise considered in physics that we are not able to treat mathematically.

Proposition C.3.2 *Let T and R be positive numbers (R in $(0,4)$ for the second inequality), take D dense in $[R,R+1]$ and a sequence of operators $(\Phi_n)_{n\in\mathbb{N}}$ in $\mathcal{L}_2\left(\mathrm{L}^2,\mathrm{L}^2\right)$ such that for every $h\in\mathcal{C}_D^1$ we have $\Phi_n h$ converges to h in $\mathrm{L}^1\left(0,T;\mathrm{L}^2\right)$. Then we obtain*

$$\underline{\lim}_{n\to\infty,\epsilon\to 0}\epsilon\log\mathbb{P}\left(\mathbf{N}\left(u^{\epsilon,0,n}(T)\right)\geq R\right)\geq -\frac{R(12+\pi^2)}{18T}.$$

Replacing in the above $\mathcal{C}_D^1$ by $\mathcal{C}_D^2$ we obtain

$$\underline{\lim}_{n\to\infty,\epsilon\to 0}\epsilon\log\mathbb{P}\left(\mathbf{N}\left(u^{\epsilon,\Psi,n}(T)\right)-4<-R\right)\geq -\frac{2(8-R-4\sqrt{4-R})(12+\pi^2)}{36T}.$$

The exponent n is there to recall that Φ is replaced by Φ_n,

Note that the result in Proposition C.3.1 depends on Φ only through its norm as a bounded operator in L^2. It is not difficult to see that there exists sequences of operators $(\Phi_n)_{n\in\mathbb{N}}$ satisfying the assumptions of Proposition C.3.2, *i.e.* which are Hilbert-Schmidt from L^2 to L^2 and Φ_n approximates the identity on the good set of controls, and are uniformly bounded as operators on L^2 by a constant independent on T. For such sequences of operators, the upper and lower bounds given above agree up to constants in their behavior in large T.

It is obtained in [63], for the ideal white noise and using the heuristic arguments recalled in the introduction, that the probability density function of the amplitude of the pulse at coordinate T when the initial datum is null is asymptotically that of an exponential law of parameter $\frac{\epsilon T}{2}$. The amplitude is a constant times the mass for the modulated soliton solutions considered [63]. Integrating this density over $[\frac{R}{2}, \infty)$ and taking into account the different normalisation, we obtain $\lim_{\epsilon \to 0} \epsilon \log \mathbb{P}\left(\mathbf{N}\left(u^{\epsilon,0}(T)\right) \geq R\right) = -\frac{R}{T}$. It is in between our two bounds and very close to our lower bound. A surprising fact is that, we obtain our result by parameterizing only the amplitude whereas in [63] a much more general parametrization is used. Both bounds exhibit the right behavior in R and T. Moreover, the order in R confirms physical and numerical results that the law is not Gaussian. On a log scale the order in R is that of tails of an exponential law. In such a case the Gaussian approximation leads to incorrect tails and error estimates.

Let us now comment on our results in the case of a soliton as initial datum. In [63], the error probability when the size of the measurement window is of the order of the coordinate T is obtained. It is given by $\lim_{\epsilon \to 0} \epsilon \log \mathbb{P}\left(\mathbf{N}\left(u^{\epsilon,\Psi}(T)\right) - 4 < -R\right) = -\frac{c(R)}{T}$, with a constant $c(R)$. It exhibits the same behavior in T as in our calculations. The discussion on the behavior with respect to R is less clear. Our bounds are not of the same order. In [51, 110] the PDF of the mass at coordinate T for a soliton profile as initial datum is not Gaussian. The numerical simulations in [114] also exhibit a significant difference between the log of the tails of the amplitude and that of a Gaussian law. Our lower bound indicates that again the tails are larger than Gaussian tails. Thus we give a rigorous proof of the fact that a Gaussian approximation is incorrect.

Finally, it is natural to obtain that the tails of the mass are increasing functions of T since the higher is T, the less energy is needed to form a signal whose mass gets above a fixed threshold at T. Replacing above by under, the same holds in the case of a soliton as initial datum.

Remark C.3.3 *The* H^1 *setting is not required here. We could as well work with* L^2 *solutions and a LDP in* L^2*. However, it is required to work in* H^1 *for the study of the center below.*

We now estimate the tails of the center. As for the mass, the rate is hard to handle since it involves an optimal control problem for controlled NLS equations. We again deduce the asymptotic of the tails from the LDP looking at upper and lower bounds. We consider that the initial datum is Ψ_A^0 since only in this case the timing jitter might be troublesome.

Let us begin with an upper bound. It is deduced from the equation of motion of the center in the controlled NLS equation (C.2.9).

Proposition C.3.4 *For every positive T, A and R and every operator Φ in $\mathcal{L}_2\left(\mathrm{L}^2,\Sigma\right)$, the following inequality holds*

$$\overline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(\mathbf{Y}\left(u^{\epsilon,\Psi_A^0}(T)\right)\geq R\right)\leq -\frac{R^2}{8T(2T+1)^2\left(4A+\frac{R}{2T+1}\right)\|\Phi\|^2_{\mathcal{L}_c(\mathrm{L}^2,\Sigma)}}.$$

Proof. Differentiating the momentum of the solution with respect to time and replacing the time derivative of the solution with the corresponding terms of the equation we obtain

$$\left[\mathbf{P}\left(\mathbf{S}^{a,\Psi_A^0}(h)(t)\right)\right]'=4\mathfrak{Re}\int_{\mathbb{R}}\mathbf{S}^{a,\Psi_A^0}(h)(t,x)\left(\partial_x\overline{\Phi h}\right)(t,x)dx.$$

Indeed by successive integration by parts all terms cancel besides the one involving the forcing term. Since $\mathbf{Y}\left(\Psi_A^0\right)=0$ and $\mathbf{P}\left(\Psi_A^0\right)=0$, thanks to (C.2.9), we obtain the identity

$$\begin{aligned}\mathbf{Y}(\mathbf{S}^{a,\Psi_A^0}(h)(t))= &\ 4\mathfrak{Re}\left(\int_0^t\int_0^s\int_{\mathbb{R}}\overline{\mathbf{S}^{a,\Psi_A^0}(h)(\sigma,x)}\left(\partial_x\Phi h\right)(\sigma,x)dxd\sigma ds\right)\\ &-2\mathfrak{Re}\left(i\int_0^t\int_{\mathbb{R}}x\overline{\mathbf{S}^{a,\Psi_A^0}(h)(s,x)}\left(\Phi h\right)(s,x)dxds\right).\end{aligned}$$

From this identity it follows that the controls h in the minimizing set of the LDP applied to the event we consider necessarily satisfy

$$\begin{aligned}R\leq\mathbf{Y}\left(\mathbf{S}^{a,\Psi_A^0}(h)(T)\right)\leq &\ 4T\|\Phi\|_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{H}^1)}\|h\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}\|\mathbf{S}^{a,\Psi_A^0}(h)\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}\\ &+2\|\Phi\|_{\mathcal{L}_c(\mathrm{L}^2,\Sigma)}\|h\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}\|\mathbf{S}^{a,\Psi_A^0}(h)\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}.\end{aligned}$$

Moreover, arguing as in the proof of Proposition C.3.1, see also [81],

$$\begin{aligned}\|\mathbf{S}^{a,\Psi_A^0}(h)\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}\leq &\ T\|\Phi\|_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{L}^2)}\|h\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}\\ &\left(1+\sqrt{1+\frac{4A}{T\|\Phi\|^2_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{L}^2)}\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)}}}\right).\end{aligned}$$

A lower bound on $\frac{1}{2}\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)}$ follows easily since the function $x\mapsto x\left(1+\sqrt{1+\frac{4}{x}}\right)$ is increasing on $\mathbb{R}_+^*$. The result follows. □

A lower bound is obtained considering controls suggested at the end of Section 2 and minimizing on the smaller set of controls. We define the following set of control for A, T positive and D a subset of $(0,\infty)$

$$\begin{aligned}\mathcal{H}^D_{A,T}= &\ \{h\in\mathrm{L}^2(0,T;\mathrm{L}^2),\ h(t,x)=\lambda(t)\left(x-2\int_0^t\int_0^s\lambda(\tau)d\tau ds\right)\tilde{\Psi}_{A,\lambda}(t,x),\\ &\qquad\text{with }\lambda(t)=\tfrac{3\tilde{R}(T-t)}{8AT^3},\ \tilde{R}\in D\}\end{aligned}$$

Proposition C.3.5 *Let T, A and R be positive. Assume that, for a dense set D of $[R, R+1]$, $(\Phi_n)_{n\in\mathbb{N}}$ is a sequence of operators in $\mathcal{L}_2\left(\mathrm{L}^2, \Sigma\right)$ such that for any h in $\mathcal{H}_{T,A}^D$, $\Phi_n h$ converges to h in $\mathrm{L}^1(0,T;\Sigma)$. Then we have the following inequality where the n in the exponent recalls that Φ is replaced by Φ_n,*

$$\underline{\lim}_{n\to\infty,\epsilon\to 0}\epsilon \log \mathbb{P}\left(\mathbf{Y}\left(u^{\epsilon,\Psi_A^0,n}(T)\right) \geq R\right) \geq -\frac{\pi^2 R^2}{128 T^3 A^3}.$$

Proof. By the LDP for the center $\mathbf{Y}$, we know that for a fixed n a lower bound is given by

$$-\inf_{y>R} I_{Y,n}^{\Psi_A^0}(y)$$

where

$$I_{Y,n}^{\Psi_A^0}(y) = \frac{1}{2} \inf_{h\in \mathrm{L}^2(0,T;\mathrm{L}^2):\ \mathbf{Y}\left(\mathbf{S}^{a,\Psi_A^0,n}(h)(T)\right)=y} \left\{\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)}\right\}.$$

Again, the n is there to recall that in the skeleton equation, Φ is replaced by Φ_n. To minorize this quantity, we first treat the case $\Phi = I$. Note that the stochastic equation has no meaning in this case but the skeleton equation has a well defined solution provided $h \in \mathrm{L}^2(0,T;\mathrm{L}^2)$. We denote by $\mathbf{S}_{WN}^{a,\Psi_A^0}$ the skeleton when $\Phi = I$. It is not difficult to see that $\mathbf{S}_{WN}^{a,\Psi_A^0}(h)$ belongs to $\mathrm{L}^\infty([0,T];\Sigma)$ when h belong to $\mathrm{L}^1(0,T;\Sigma)$. A standard argument to prove this is to compute the second derivative with respect to time of the variance $\mathbf{V}(u) = \int_{\mathbb{R}} x^2|u(t,x)|^2 dx$ when $u = \mathbf{S}_{WN}^{a,\Psi_A^0}(h)$. It is also standard to prove that, for each t, the mapping $h \to \mathbf{S}_{WN}^{a,\Psi_A^0}(h)(t)$ is weakly continuous from $\mathrm{L}^1(0,T;\Sigma)$ to Σ and strongly continuous from $\mathrm{L}^1(0,T;\Sigma)$ to H^1 . Therefore, since Y is weakly continuous on Σ, thanks to our assumptions, we know that for $h \in \mathcal{H}_{T,A}^D$

$$\mathbf{Y}\left(\mathbf{S}^{a,\Psi_A^0,n}(h)(T)\right) = \mathbf{Y}\left(\mathbf{S}_{WN}^{a,\Psi_A^0}(\Phi_n h)(T)\right) \to \mathbf{Y}\left(\mathbf{S}_{WN}^{a,\Psi_A^0}(h)(T)\right) \text{ when } n\to\infty. \tag{C.3.3}$$

Let $\tilde{\mathcal{H}}_{T,A}$ be the same set of controls as above but where λ is only assumed to belong to $\mathrm{L}^2(0,T;\mathbb{R})$

$$\tilde{\mathcal{H}}_{T,A} = \{h \in \mathrm{L}^2(0,T;\mathrm{L}^2),\ h(t,x) = \lambda(t)\left(x - 2\int_0^t\int_0^s \lambda(\tau)d\tau ds\right)\tilde{\Psi}_{A,\lambda}(t,x), \\ \lambda \in \mathrm{L}^2(0,T;\mathbb{R})\}.$$

Clearly,

$$\begin{aligned}
&\inf_{h\in \mathrm{L}^2(0,T;\mathrm{L}^2):\ \mathbf{Y}\left(\mathbf{S}_{WN}^{a,\Psi_A^0}(h)(T)\right)\geq \tilde{R}} \|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)}\\
&\leq \inf_{h\in \tilde{\mathcal{H}}_{T,A}:\ \mathbf{Y}\left(\mathbf{S}_{WN}^{a,\Psi_A^0}(h)(T)\right)\geq \tilde{R}} \|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)}\\
&= \inf_{\lambda\in \mathrm{L}^2(0,T;\mathbb{R}),\int_0^T\int_0^t \lambda(s)dsdt\geq \frac{\tilde{R}}{8A}} \frac{\pi^2}{3A}\int_0^T \lambda^2(t)dt
\end{aligned}$$

Note that the contraint $\int_0^T\int_0^t \lambda(s)dsdt \geq \frac{\tilde{R}}{8A}$, is not a boundary condition as in the usual calculus of variations. To solve this minimization problem, we use the quantity $\mathcal{L}_{T,A,\tilde{R}}(\lambda)$ defined by

$$\mathcal{L}_{T,A,\tilde{R}}(\lambda) = \frac{\pi^2}{3A}\int_0^T \lambda^2(t)dt - \gamma\int_0^T\int_0^t \lambda(s)dsdt,$$

where γ belongs to $\mathbb{R}$. We then impose that our guess $\lambda^*_{T,A,\tilde{R}}$ is a critical point of $\mathcal{L}_{T,A,\tilde{R}}(\lambda)$ and that it satisfies the constraint $\int_0^T\int_0^t \lambda(s)dsdt = \frac{\tilde{R}}{8A}$. We obtain

$$\lambda^*_{T,A,\tilde{R}}(t) = \frac{3\tilde{R}(T-t)}{8AT^3}.$$

We do not claim that the minimization problem is solved, we simply write

$$\begin{aligned}
&\inf_{\lambda\in \mathrm{L}^1(0,T;\mathbb{R}),\int_0^T\int_0^t \lambda(s)dsdt\geq \frac{\tilde{R}}{8A}} \frac{\pi^2}{3A}\int_0^T \lambda^2(t)dt\\
&\leq \frac{\pi^2}{3A}\int_0^T \lambda^*_{T,A,\tilde{R}}(t)dt = \frac{\pi^2\tilde{R}^2}{64A^3T^3}
\end{aligned}$$

Let us set

$$h^*_{\tilde{R}}(t,x) = \lambda^*_{T,A,\tilde{R}}(t)\left(x - 2\int_0^t\int_0^s \lambda^*_{T,A,\tilde{R}}(\tau)d\tau ds\right)\tilde{\Psi}_{A,\lambda^*_{T,A,\tilde{R}}}(t,x).$$

By (C.3.3), we have for $\tilde{R}\in D$,

$$\mathbf{Y}\left(\mathbf{S}^{a,\Psi_A^0,n}(h^*_{\tilde{R}})(T)\right) \to \mathbf{Y}\left(\mathbf{S}_{WN}^{a,\Psi_A^0}(h^*_{\tilde{R}})(T)\right) \text{ when } n\to\infty.$$

Therefore, for n large enough,

$$\mathbf{Y}\left(\mathbf{S}^{a,\Psi_A^0,n}(h^*_{\tilde{R}})(T)\right) > R.$$

We deduce

$$\inf_{x>R} I_{Y,n}^{\Psi_A^0}(x) \leq \frac{\pi^2 \tilde{R}^2}{64A^3T^3}.$$

Since this is true for $\tilde{R}$ in a dense set of $[R, R+1]$ we deduce the result. $\square$

The upper and lower bounds given in Proposition C.3.4 and C.3.5 are in perfect agreement in their behavior with respect to R and to T when T is large. Indeed, for T large, the upper bound in Proposition C.3.4 is close to $\frac{R^2}{128T^3A\|\Phi\|_{\mathcal{L}_c(\mathrm{L}^2,\Sigma)}}$. However, we have to be careful before doing such a comparison. Indeed, the bounds can be compared only if we are able to consider a sequence of operators $(\Phi_n)_{n\in\mathbb{N}}$ satisfying the assumptions of Proposition C.3.5 and such that $\|\Phi_n\|_{\mathcal{L}_c(\mathrm{L}^2,\Sigma)}$ is bounded uniformly in n.

It seems possible to construct such a sequence. For instance we may choose $\widetilde{\Phi}$ in $\mathcal{L}_2(\mathrm{L}^2, \Sigma)$ such that $\widetilde{\Phi}k = k$ for k in K_A, the closure in L^2 of the vector space spanned by $\{(x-a)\mathrm{sech}\,(A(x-b)),\ a \in [0,1],\ b \in [0,1]\}$. We believe that K_A is embedded in Σ in a Hilbert-Schmidt way. For T and A sufficiently large and $D \subset [R, R+1]$, each h in the set $\mathcal{H}_{A,T}^D$ is such that $h(t) \in K_A$ for $t \in [0,T]$, thus $\widetilde{\Phi}h = h$ and we can take $\Phi_n = \widetilde{\Phi}$ in Proposition C.3.5. In this case, the two bounds are comparable and are of the same order in R and T. Note that $\|\Phi_n\|_{\mathcal{L}_c(\mathrm{L}^2,\Sigma)}$ is independent on R and T.

In fact, many such sequences probably exist. Therefore, it seems that the bounds can be compared in many circumstances. Roughly speaking, the fact that this can be done means that we are treating noises which are sufficiently localized around the soliton Ψ_A^0.

If the sequence $(\Phi_n)_{n\in\mathbb{N}}$ converges pointwise to the identity, *i.e.* if we wish to understand what happens in the white noise limit, then this localization assumption does not hold. In this case, the lower bound is meaningful whereas the upper bound converges to zero and provides no information.

The comparison of the behavior of the bounds with respect to A is less clear. The two bounds seem contradictory for large A. This is due to the fact that it is not possible to choose a sequence of operators $(\Phi_n)_{n\in\mathbb{N}}$ satisfying the assumptions of Proposition C.3.5 and such that $\|\Phi_n\|_{\mathcal{L}_c(\mathrm{L}^2,\Sigma)}$ is uniformly bounded with respect to A. Indeed such a sequence necessarily satisfies

$$\|h\|_{\mathrm{L}^1(0,T;\Sigma)} \leq \underline{\lim}_{n\to\infty} \|\Phi_n\|_{\mathcal{L}_c(\mathrm{L}^2,\Sigma)} \|h\|_{\mathrm{L}^1(0,T;\mathrm{L}^2)}$$

for any $h \in \mathcal{H}_{T,A}^D$. It is easily seen that for A and T sufficiently large, the ratio of $\|h\|_{\mathrm{L}^1(0,T;\Sigma)}$ and $\|h\|_{\mathrm{L}^1(0,T;\mathrm{L}^2)}$ is of the order A.

In fact this shows that the upper bound in Proposition C.3.4 is always larger than a constant times $\frac{R^2}{T^3A^3}$ for a sequence satisfying the assumptions of Proposition C.3.5. Thus there is no contradiction.

We can probably go further. Indeed, there may exist sequences of operators satisfying the assumptions of Proposition C.3.5 and such that $\|\Phi_n\|_{\mathcal{L}_c(\mathrm{L}^2,\Sigma)} \leq cA$ for some constant c. In this case the bounds are of the same order with respect to A, R and T. An example could be constructed in the same way as above. It suffices to take Φ_n equal to the identity on K_A and zero on a complementary space. Indeed, it can be shown that $\|h\|_\Sigma \leq cA\|h\|_{\mathrm{L}^2}$ for some constant c.

Therefore, the two bounds are also comparable in their behavior with respect to A under a localization assumption on the noise.

Let us now compare our result with the results obtained in the physics literature. First, we note that we obtain that on a log scale the tails are equivalent to Gaussian tails. This is indeed the kind of result obtained by arguments from the physical theory of perturbation of solitons.

Remark C.3.6 *We are missing the pre exponential factors to conclude whether or not the tails are Gaussian. We could think of using sharp Laplace asymptotics to obtain these factors.*

Now, suppose the law were indeed Gaussian, then the asymptotic of the tails may be written in terms of the variance. By doing so, we find that the variance of the timing jitter is of the order T^3. It agrees perfectly with the initial results of [88]. Also the order in both A and T seems to agree perfectly with the orders of the contribution of the additive noise to the variance of the timing jitter in equation (3.18) in [59]. Note however that in [88, 97], where the model is instead a juxtaposition of deterministic evolutions with random initial data in between amplifiers, the order in A seems to be $-\frac{c}{A}$.

We end this section noticing that our result confirms the fact that, in the presence of additive noise, the timing jitter is more troublesome than the fluctuation of the mass when we consider the problem of losing a signal. Indeed we have found that the error probability due to timing jitter is of the order of $\exp\left(-\frac{c_1(R)}{\epsilon T^3}\right)$ and an error probability due to the fluctuation of the mass is of the order of $\exp\left(-\frac{c_2(R)}{\epsilon T}\right)$ which is clearly negligible compared to the first for large T. Recall that T represents the length of a fiber optical line and is thus assumed to be very large.

Remark C.3.7 *From an engineering point of view it is possible to exponentially reduce the probability of undesired deviations of the center by introducing inline control elements; see for example [63]. We could also use ideas given in [128] and optimize on such external fields for a limited cost* *or penalty functional. The new optimal control problem requires then double optimization.*

Remark C.3.8 *Note that the methodology developed herein could also be applied to the determination of the small noise asymptotic of the tails of the position of an isolated vortex, defined by* $\oint \nabla \arg u(t,x) \cdot d\mathbf{l}$*, in a Bose condensates or superfluid Helium as in [125]. There the physical perturbation approach along with the Fokker-Planck equation are used. The small noise acts as the small temperature.*

C.4 Tails of the center in the multiplicative case

In the case of the multiplicative noise, the mass is a conserved quantity and we restrict our attention to the case of the law of the center of the pulse when the initial datum is the soliton profile Ψ_A^0.

Again, let us begin with upper bounds obtained from an equation for the motion of the center in the controlled NLS equation.

From relation (C.2.8) and integration by parts, we obtain the equation in [141],

$$\left[\mathbf{Y}(\mathbf{S}^{m,\Psi_A^0}(h)(t))\right]'' = 2\int_{\mathbb{R}} |\mathbf{S}^{m,\Psi_A^0}(h)(t,x)|^2 \left(\partial_x \Phi h\right)(t,x)dx. \tag{C.4.1}$$

We may thus deduce the next proposition.

Proposition C.4.1 *For every positive T, A and R and every operator Φ in $\mathcal{L}_2\left(\mathrm{L}^2, \mathrm{H}^s(\mathbb{R},\mathbb{R})\right)$, where $s > \frac{3}{2}$ the following inequality holds*

$$\overline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(\mathbf{Y}\left(u^{\epsilon,\Psi_A^0}(T)\right) \geq R\right) \leq -\left(\frac{3}{16}\right)^2 \frac{R^2}{2A^2T^3\|\Phi\|^2_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{W}^{1,\infty}(\mathbb{R},\mathbb{R}))}}.$$

Proof. From equation (C.4.1), the fact that $\mathbf{Y}\left(\mathbf{S}^{m,\Psi_A^0}(h)\right)'(0) = \mathbf{P}(\Psi_A^0) = 0$, that for such values of s the injection of $\mathrm{H}^s(\mathbb{R},\mathbb{R})$ into $\mathrm{W}^{1,\infty}(\mathbb{R},\mathbb{R})$ is continuous and that the mass is conserved and thus remains equal to 4 we obtain that

$$\begin{aligned}\mathbf{Y}\left(\mathbf{S}^{m,\Psi_A^0}(h)(t)\right)' &\leq 8A\|\Phi\|_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{W}^{1,\infty}(\mathbb{R},\mathbb{R}))}\|h\|_{\mathrm{L}^1(0,t;\mathrm{L}^2)} \\ &\leq 8A\sqrt{t}\|\Phi\|_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{W}^{1,\infty}(\mathbb{R},\mathbb{R}))}\|h\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}\end{aligned}$$

Then, since $\mathbf{Y}\left(\Psi_A^0\right)=0$, we obtain integrating the above inequality that

$$R \leq \mathbf{Y}\left(\mathbf{S}^{m,\Psi_A^0}(h)(T)\right) \leq \frac{16AT^{\frac{3}{2}}}{3}\|\Phi\|_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{W}^{1,\infty}(\mathbb{R},\mathbb{R}))}\|h\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}$$

and the conclusion follows. $\square$

Let us consider now lower bounds. We need to find controls which have the desired effect on the center. We have seen that in the additive case, good controls are given by functions in $\mathcal{H}_{A,T}^D$. Recalling the transformations on the equation made at the end of Section 2, we can equivalently take controls of the form $\lambda(t)x\Psi_{A,\lambda}$ which correspond to the solution $\Psi_{A,\lambda}$. Thus, in the multiplicative case, a good control is given by $h(t,x)=\lambda(t)x$. Unfortunately these controls do not belong to the range of Φ nor to $\mathrm{L}^2\left(0,T;\mathrm{L}^2\right)$ and are not admissible.

We have tried to approximate these controls by admissible ones. Since the control is multiplied by $\Psi_{A,\lambda}$ in the equation, it seems that it has no effect outside a set centered around the center of $\Psi_{A,\lambda}$ and that we could replace $\lambda(t)x$ by a truncation. We have not been able to get any information by such arguments. We have tried several other choices of control corresponding to various modulated solitons especially with a phase nonlinear in x. They never yielded the right order of the lower bound with respect to A or T. We therefore impose a new assumption that Φ takes its values in $\mathrm{H}^s(\mathbb{R},\mathbb{R})\oplus x\mathrm{L}^1(0,T;\mathbb{R})$. In other words we consider the slightly different equation

$$i\mathrm{d}\tilde{u}^{\epsilon,u_0}=\left(\Delta\tilde{u}^{\epsilon,u_0}+|\tilde{u}^{\epsilon,u_0}|^2\tilde{u}^{\epsilon,u_0}\right)\mathrm{d}t+\tilde{u}^{\epsilon,u_0}\circ\sqrt{\epsilon}\mathrm{d}W(t)+\sqrt{\epsilon}x\tilde{u}^{\epsilon,u_0}\circ\mathrm{d}\beta(t) \tag{C.4.2}$$

where β is a standard Brownian motion independant of W and the corresponding controlled PDE

$$\begin{aligned} i\tfrac{\mathrm{d}}{\mathrm{d}t}\tilde{S}^{u_0}(h_1,h_2)= \quad & \Delta\tilde{S}^{u_0}(h_1,h_2)+|\tilde{S}^{u_0}(h_1,h_2)|^2\tilde{S}^{u_0}(h_1,h_2)\\ & +\tilde{S}^{u_0}(h_1,h_2)\Phi h_1+x\tilde{S}^{u_0}(h_1,h_2)h_2 \end{aligned}$$

where h_1 belongs to $\mathrm{L}^2\left(0,T;\mathrm{L}^2\right)$ and h_2 belongs to $\mathrm{L}^2(0,T;\mathbb{R})$, the initial datum is u_0 and in the sequel $u_0=\Psi_A^0$. We may guess by successive applications of the Itô formula, multiplying $\tilde{u}^{\epsilon,u_0}$ by the random phase term $\exp\left(ix\sqrt{\epsilon}\beta(t)\right)$, and similar transformations as in Section 2 (stochastic gauge transform, stochastic methods of characteristics...) that we should consider the function

$$\exp\left(ix\sqrt{\epsilon}\beta(t)-i\epsilon\int_0^t\beta^2(s)ds\right)\tilde{u}^{\epsilon,u_0}\left(t,x+2\sqrt{\epsilon}\int_0^t\beta(s)ds\right).$$

It indeed satisfies equation (C.2.2) with same initial datum. We deduce that

$$\tilde{u}^{\epsilon,u_0}(t,x) =$$
$$\exp\left(-ix\sqrt{\epsilon}\beta(t) + i\epsilon\int_0^t \beta^2(s)ds + 2i\epsilon\beta(t)\int_0^t \beta(s)ds\right) u^{\epsilon,u_0}\left(t, x - 2\sqrt{\epsilon}\int_0^t \beta(s)ds\right).$$

A similar computation shows that

$$\tilde{S}^{u_0}(h_1,h_2)(t,x) = \exp\left(-ix\sqrt{\epsilon}\int_0^t h_2(s)ds + i\int_0^t \left(\int_0^s h_2(u)du\right)^2 ds\right.$$
$$\left. + 2i\int_0^t h_2(s)ds\int_0^t\int_0^s h_2(u)duds\right) \mathbf{S}^{m,u_0}(h_1)\left(t, x - 2\int_0^t\int_0^s h_2(u)du\right).$$

The functions $\tilde{u}^{\epsilon,u_0}$ and $\tilde{S}^{u_0}(h_1,h_2)$ are well defined functions of $\mathrm{L}^2(0,T;\Sigma)$ and we may compute their centers. We obtain a lower bound of the asymptotic of the tails of the center of the new solutions.

Proposition C.4.2 *For every positive T, A and R and every operator Φ in*
$\mathcal{L}_2\left(\mathrm{L}^2, \mathrm{H}^s(\mathbb{R},\mathbb{R})\right)$ *where $s > \frac{3}{2}$ the following inequality holds*

$$\underline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(\mathbf{Y}\left(u^{\epsilon,\Psi_A^0}(T)\right) \geq R\right) \geq -\frac{3R^2}{128A^2T^3}.$$

Proof. Consider the mapping F from $\mathrm{C}\left([0,T];\Sigma^{\frac{1}{2}}\right) \times \mathrm{C}([0,T];\mathbb{R})$ into $\mathbb{R}$ such that

$$F(u,b) = \int_{\mathbb{R}} |x| \left|u\left(T, x - 2\int_0^T b(s)ds\right)\right|^2 dx.$$

Take u and u' in $\mathrm{C}\left([0,T];\Sigma^{\frac{1}{2}}\right)$ and b and b' in $\mathrm{C}([0,T];\mathbb{R})$, then by the triangle and inverse triangle inequalities and the change of variables we obtain

$$\begin{aligned}
&|F(u,b) - F(u',b')| \\
&\leq \int_{\mathbb{R}} \left|\left|x + 2\int_0^T b(s)ds\right| - \left|x + 2\int_0^T b'(s)ds\right|\right| |u(T,x)|^2 dx \\
&\quad + \left|\int_{\mathbb{R}} \left|x + 2\int_0^T b'(s)ds\right| \left(|u(T,x)|^2 - |u'(T,x)|^2\right) dx\right| \\
&\leq 2\left|\int_0^T b(s)ds - \int_0^T b'(s)ds\right| \int_{\mathbb{R}} |u(T,x)|^2 dx \\
&\quad + \int_{\mathbb{R}} |x| \, ||u(T,x)| - |u'(T,x)|| \left(|u(T,x)| + |u'(T,x)|\right) dx \\
&\quad + 2\left|\int_0^T b'(s)ds\right| \int_{\mathbb{R}} ||u(T,x)| - |u'(T,x)|| \left(|u(T,x)| + |u'(T,x)|\right) dx
\end{aligned}$$

we conclude from the inverse triangle and Hölder inequalities that F is continuous. We may then push forward the LDP for the paths of u^{ϵ,Ψ_A^0} and of the Brownian motion by the mapping F using a slight modification of the result of exercise 4.2.7 of [48] and obtain a LDP for the laws of $\mathbf{Y}\left(\tilde{u}^{\epsilon,\Psi_A^0}(T)\right)$ which is that of $F\left(u^{\epsilon,\Psi_A^0},\sqrt{\epsilon}\beta\right)$ of speed ϵ and good rate function defined as a function of the rate function of the original solutions and of the rate function I_β of the sample path LDP for the Brownian motion

$$\begin{array}{rl}\tilde{I}_Y^{\Psi_A^0}(x)= & \inf_{(u,b):\ F(u,b)=x}\left(I^{u_0}(u)+I_\beta(b)\right)\\ \leq & \frac{1}{2}\inf_{(h_1,h_2):\ F\left(\mathbf{S}^{m,\Psi_A^0}(h_1),\int_0^\cdot h_2(s)ds\right)=x}\left\{\|h_1\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)}+\|h_2\|^2_{\mathrm{L}^2(0,T;\mathbb{R})}\right\}\\ \leq & \frac{1}{2}\inf_{(h_1,h_2):\ \mathbf{Y}\left(\tilde{\mathbf{S}}^{\Psi_A^0}(h_1,h_2)(T)\right)=x}\left\{\|h_1\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)}+\|h_2\|^2_{\mathrm{L}^2(0,T;\mathbb{R})}\right\}.\end{array}$$

Thus considering solely controls of the from $(0,h_2)$, we minimize in h_2 for γ in $\mathbb{R}$,

$$\int_0^T h_2^2(t)dt-\gamma\int_0^T\int_0^t h_2(s)ds,$$

where we impose that

$$\mathbf{Y}\left(\Psi_{A,h_2}(T)\right)=8A\int_0^T\int_0^t h_2(s)ds=\tilde{R}>R.$$

The conclusion follows. □

Remark C.4.3 *We may check that* $\mathbf{Y}\left(u^{\epsilon,\Psi_A^0}\right)=\mathbf{Y}\left(\tilde{u}^{\epsilon,\Psi_A^0}\right)-8\sqrt{\epsilon}\int_0^T\beta(s)ds$ *and that* $\int_0^T\beta(s)ds$ *is a centered Gaussian random variable with variance* $\frac{T^3}{3}$*.*

The corresponding upper bound for this modified stochastic NLS equation is

$$\overline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(\mathbf{Y}\left(u^{\epsilon,\Psi_A^0}(T)\right)\geq R\right)\leq-\left(\frac{3}{16}\right)^2\frac{R^2}{A^2T^3\left(\|\Phi\|^2_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{W}^{1,\infty}(\mathbb{R},\mathbb{R}))}\vee 1\right)}.$$

Note that the lower bound do not require to consider a sequence of operators $(\Phi_n)_{n\in\mathbb{N}}$ and we may indeed compare the upper and lower bounds. They are of the same order in T and in A. Note also that, as in the additive case, we obtain that on a log scale the tails are equivalently that of Gaussian tails. Also, our tails are of the order in T that we expect from the contribution of the multiplicative noise to the variance of the timing jitter in equation (3.18) in [59].

However, concerning the amplitude, it is not of the order of $-\frac{c}{A^4}$ as we would expect from [59]. This is probably due to the fact that we have considered a colored noise with a term $x\frac{\mathrm{d}}{\mathrm{dt}}\beta$ that grows linearly in time (the x variable). We have explained that, otherwise, we fail to obtain a lower bound. We have obtained, that for large A, and thus for even more localized in time solitons, the tails of the center in the additive noise are larger than that in the multiplicative noise. Note that it is predicted in [59] that the quantum Raman noise is a dominant source of fluctuations in phase and arrival time for sub-picosecond solitons and that on the other hand for longer solitons, Raman effects are reduced compared to the usual Gordon-Haus jitter. It seems at first glance to be in contradiction with our results but their result is obtained for $A = 1$ and time corresponds to the typical pulse duration considered for scaling purposes in order to obtain the NLS equation; also our order in A differs from theirs.

C.5 Annex - proof of Theorem C.2.1

We denote herein by $\mathbf{V}(f) = \int_{\mathbb{R}} |x|^2 |f(x)|^2 dx$ the variance defined for f in Σ.

Let us start with the additive case. We denote by $v^{u_0}(z)$ the solution of

$$\begin{cases} i\frac{\mathrm{d}v}{\mathrm{dt}} = \Delta v + \lambda |v - iz|^{2\sigma}(v - iz) \\ u(0) = u_0 \in \Sigma \end{cases},$$

where z belongs to $X^{(T,2\sigma+2,\Sigma)} = \mathrm{C}([0,T];\Sigma) \cap \mathrm{L}^r\left(0,T;\mathrm{W}^{1,2\sigma+2}\right)$ and r is such that $\frac{2}{r} = \frac{1}{2} - \frac{1}{2\sigma+2}$. We also denote by $\mathcal{G}^{u_0}$ the mapping

$$z \mapsto v^{u_0}(z) - iz,$$

it is such that $u^{\epsilon,u_0} = \mathcal{G}^{u_0}(\sqrt{\epsilon}Z)$ where Z is the stochastic convolution defined by $Z(t) = \int_0^t U(t-s)dW(s)$.
We can check from similar arguments as those of the proof of Proposition 1 in [81] that the stochastic convolution is a $X^{(T,2\sigma+2,\Sigma)}$ random variable whose law μ^Z is a centered Gaussian measure. Let z belong to $X^{(T,2\sigma+2,\Sigma)}$, take $s < t < T$, the triangle along with the Hölder inequalities then allow to compute

$$\begin{aligned}
&\left|\int_{\mathbb{R}} |x| \left(|\mathcal{G}^{u_0}(z)(t,x)|^2 - |\mathcal{G}^{u_0}(z)(s,x)|^2\right) dx\right| \\
&\leq \int_{\mathbb{R}} |x|(|\mathcal{G}^{u_0}(z)(t,x)| + |\mathcal{G}^{u_0}(z)(s,x)|)|(|\mathcal{G}^{u_0}(z)(t,x)| - |\mathcal{G}^{u_0}(z)(s,x)|)|dx \\
&\leq \|\mathcal{G}^{u_0}(z)(t) - \mathcal{G}^{u_0}(z)(s)\|_{\mathrm{L}^2} \sqrt{\mathbf{V}(|\mathcal{G}^{u_0}(z)(t)| + |\mathcal{G}^{u_0}(z)(s)|)} \\
&\leq 2\sqrt{2}\|\mathcal{G}^{u_0}(z)(t) - \mathcal{G}^{u_0}(z)(s)\|_{\mathrm{L}^2} \\
&\quad \times \left(\sqrt{\mathbf{V}(v^{u_0}(z)(t))} + \sqrt{\mathbf{V}(v^{u_0}(z)(s))} + \sqrt{\mathbf{V}(z(t))} + \sqrt{\mathbf{V}(z(s))}\right).
\end{aligned}$$

The application of the Gronwall inequality in the proof of Proposition 3.5 in [39], along with the Sobolev injection allow to prove that $\mathcal{G}^{u_0}(z)$ belongs to $\mathrm{C}([0,T];\Sigma^{\frac{1}{2}})$. The computation above also shows that the mapping $\mathcal{G}^{u_0}$ is continuous from $X^{(T,2\sigma+2,\Sigma)}$ to $\mathrm{C}([0,T];\Sigma^{\frac{1}{2}})$. The general result on LDP for Gaussian measures gives the LDP for the measures μ^{Z_ϵ}, the direct images of μ^Z under the transformation $x \mapsto \sqrt{\epsilon}x$ on $X^{(T,2\sigma+2,\Sigma)}$. We conclude with the contraction principle.

In the multiplicative case, it is also required to revisit the proof of the LDP in [82]. Note that in the following when Φh is replaced by $\frac{\partial f}{\partial t}$ where f belongs to $\mathrm{H}_0^1\left(0,T;\mathrm{H}^s(\mathbb{R},\mathbb{R})\right)$ which is the subspace of $\mathrm{C}\left([0,T];\mathrm{H}^s(\mathbb{R},\mathbb{R})\right)$ of functions null at time 0, square integrable in time and with square integrable in time time derivative. The skeleton is then denoted by $\tilde{\mathbf{S}}^{m,u_0}(f)$.
We may check using the above calculation and the fact that for every $t \in [0,T]$, $\tilde{\mathbf{S}}^{m,u_0}(f)(t)$ belongs to Σ that

$$\mathbf{V}\left(\tilde{\mathbf{S}}^{m,u_0}(f)(t)\right) \leq \left(4\|\tilde{\mathbf{S}}^{m,u_0}(f)(t)\|^2_{\mathrm{C}([0,T];\mathrm{H}^1)} + \mathbf{V}(u_0)\right)e^T,$$

see the arguments of the proof of Proposition 3.2 in [40] used for the skeleton, that the skeleton is continuous from the sets of levels of the rate function of the Wiener process less or equal to a positive constant, with the topology induced by that of $\mathrm{C}\left([0,T];\mathrm{H}^s(\mathbb{R},\mathbb{R})\right)$, to $\mathrm{C}\left([0,T];\Sigma^{\frac{1}{2}}\right)$. The only difference in the proof of Proposition 4.1 in [82], the Azencott lemma (also called Freidlin-Wentzell inequality or almost continuity of the Itô map) is in step 2. It is the reduction to estimates on the stochastic convolution. We use

$$\mathbf{V}\left(v^{\epsilon,\tilde{u}_0}(t)\right) \leq \left(4\|v^{\epsilon,\tilde{u}_0}(t)\|^2_{\mathrm{C}([0,T];\mathrm{H}^1)} + \mathbf{V}(\tilde{u}_0)\right)e^T,$$

see the proof of Proposition 3.2 in [40], where $v^{\epsilon,\tilde{u}_0}$ satisfies $v^{\epsilon,\tilde{u}_0}(0) = \tilde{u}_0$ and

$$idv^{\epsilon,\tilde{u}_0} = \left(\Delta v^{\epsilon,\tilde{u}_0} + \lambda|v^{\epsilon,\tilde{u}_0}|^{2\sigma}v^{\epsilon,\tilde{u}_0} + \frac{\partial f}{\partial t}v^{\epsilon,\tilde{u}_0} - \frac{i\epsilon}{2}F_\Phi v^{\epsilon,\tilde{u}_0}\right)dt + \sqrt{\epsilon}v^{\epsilon,\tilde{u}_0}dW_\epsilon,$$

$f(\cdot) = \int_0^{\cdot}\Phi h(s)ds$, $W_\epsilon(t) = W(t) - \frac{1}{\sqrt{\epsilon}}\int_0^t \frac{\partial f}{\partial s}ds = W(t) - \frac{1}{\sqrt{\epsilon}}\int_0^t \Phi h(s)ds$, $F_\Phi(x) = \sum_{j=1}^{\infty}\left(\Phi e_j(x)\right)^2$ and $(e_j)_{j=1}^{\infty}$ is any complete orthonormal system of L^2. The bound remains the same as in [40] because of the cancelation of the extra term in the application of the Itô formula and the cancelation of the Itô-Stratonovich correction with the second order Itô correction term when the Itô formula is applied to the truncated variance $V_r(v) = \int_{\mathbb{R}}\exp(-r|x|^2)|x|^2|v(x)|^2dx$. □

Remark C.5.1 *Uniform LDPs hold (uniform with respect to initial data in balls) in the Freidlin-Wentzell formulation or compact sets in the present formulation with* $\underline{\lim}$ *and* $\overline{\lim}$*. More general nonlinearities and dimensions and the case where blow-up may occur could be considered. It is still possible to state the result in spaces of exploding paths with a projective limit topology accounting for the various integrability. Uniformity could be useful since in optical experiments the initial pulse is a laser output and it is known up to a certain level of uncertainty.*

Appendix D

Exit from a neighborhood of zero for weakly damped stochastic nonlinear Schrödinger equations

Abstract: Exit from a neighborhood of zero for weakly damped stochastic nonlinear Schrödinger equations is studied. The small noise is either complex and of additive type or real and of multiplicative type. It is white in time and colored in space. The neighborhood is either in L^2 or in H^1. The small noise asymptotic of both the first exit times and the exit points are characterized.

D.1 Introduction

The study of the first exit time from a neighborhood of an asymptotically stable equilibrium point, the exit place determination or the transition between two equilibrium points in randomly perturbed dynamical systems is important in several areas of physics among which statistical and quantum mechanics, the natural sciences, financial macro economics... The problem is relevant in nonlinear optics; see for example [100]. We shall consider the case of weakly damped nonlinear Schrödinger equations. It is a model in nonlinear optics, hydrodynamics, biology, field theory, Fermi-Pasta-Ulam chains of atoms...

For a fixed noise amplitude and for diffusions, the first exit time and the distribution of the exit points on the boundary of the domain can be characterized respectively by the Dirichlet and Poisson equations. However, when the dimension is larger than one, we may seldom solve explicitly these equations and large deviation techniques are precious tools when the noise is assumed to be small; see for example [48, 73]. The techniques used in the physics literature is often called optimal fluctuations or instanton formalism and are closely related to the large deviations.

An energy then characterizes the transition between two states and the exit from a neighborhood of an asymptotically stable equilibrium point. The energy is derived from the rate function of the sample path large deviation principle (LDP). The paths that minimize this energy are the most likely exiting paths or transitions and when the infimum is unique the system shows an almost deterministic behavior. Note that the first order of the probability are that of the Boltzman theory and the amplitude of the small noise acts as the temperature. The deterministic dynamics is sometimes interpreted as the evolution at temperature 0 and the small noise as the small temperature nonequilibrium case. In the pioneering article [67], a nonlinear heat equation perturbed by a small noise of additive type is considered. Transitions in that case are the instantons of quantum mechanics. Also in [106], predictions for a noisy exit problem are confirmed both numerically and experimentally.

We will consider weakly damped nonlinear Schrödinger equations in $\mathbb{R}^d$. Equations are perturbed by a small noise. The noise is white in time and of additive or multiplicative type. We define it as the time derivative in the sense of distributions of a Hilbert space-valued Wiener process W. The two types of noises are physically relevant; see for example [44]. When the noise is of additive type, the Hilbert space is L^2 or H^1, spaces of complex valued functions. The evolution equation is then

$$i\mathrm{d}u^{\epsilon,u_0} = (\Delta u^{\epsilon,u_0} + \lambda |u^{\epsilon,u_0}|^{2\sigma} u^{\epsilon,u_0} - i\alpha u^{\epsilon,u_0})\mathrm{d}t + \sqrt{\epsilon}\mathrm{d}W, \tag{D.1.1}$$

where α and ϵ are positive and u_0 is an initial datum in L^2 (respectively H^1). When the noise is of multiplicative type, the Hilbert space is the Sobolev space based on L^2 of real valued functions $\mathrm{H}^s_{\mathbb{R}}$ for $s > \frac{d}{2} + 1$ and the product is a Stratonovich product. In that case the equation may be written

$$i\mathrm{d}u^{\epsilon,u_0} = (\Delta u^{\epsilon,u_0} + \lambda |u^{\epsilon,u_0}|^{2\sigma} u^{\epsilon,u_0} - i\alpha u^{\epsilon,u_0})\mathrm{d}t + \sqrt{\epsilon} u^{\epsilon,u_0} \circ \mathrm{d}W. \tag{D.1.2}$$

The Wiener process W is always assumed to be colored in space since the linear group does not have global regularizing properties and is an isometry on the Sobolev spaces based on L^2. The power σ in the nonlinearity satisfies $\sigma < \frac{2}{d}$ and thus solutions do not exhibit blow-up.

In [81] and [82] we have proved sample paths LDPs for the two types of noises but without damping and deduced the asymptotic of the tails of the blow-up times. In [81] we also deduced the tails of the mass, defined later, of the pulse at the end of a fiber optical line. We have thus evaluated the error probabilities in optical soliton transmission when the receiver records the signal on an infinite time interval. In [44] we have applied the LDP to the problem of the diffusion in position of the soliton and studied the tails of the random position. Our results are in perfect agreement with results from physics obtained via heuristic arguments. The damping term in the drift here is often physically relevant but small and neglected in the models. For example in [44], in the case of an additive noise, we have considered that the gain of the amplifiers is adjusted to compensate exactly for loss and that there remains only a spontaneous emission noise.

The flow in the equations above has Hamiltonian, gradient and random components. The mass

$$\mathbf{N}(u) = \int_{\mathbb{R}^d} |u|^2 \, dx$$

characterizes the gradient component. The Hamiltonian denoted by $\mathbf{H}(u)$, defined for functions in H^1, has a kinetic and a potential term, it may be written

$$\mathbf{H}(u) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 \, dx - \frac{\lambda}{2\sigma+2}\int_{\mathbb{R}^d} |u|^{2\sigma+2} \, dx.$$

Note that the vector fields associated to the mass and Hamiltonian are orthogonal. Recall that the mass and Hamiltonian are invariant quantities of the equation without noise and damping. Other quantities like the linear or angular momentum are also invariant for nonlinear Schrödinger equations. We could rewrite, for example equation (D.1.1) as

$$\mathrm{d}u^{\epsilon,u_0} = \left(\frac{\delta \mathbf{H}(u^{\epsilon,u_0})}{\delta \overline{u^{\epsilon,u_0}}} - \frac{\alpha}{2}\frac{\delta \mathbf{N}(u^{\epsilon,u_0})}{\delta u^{\epsilon,u_0}} \right) \mathrm{d}t - i\sqrt{\epsilon}\,\mathrm{d}W.$$

Without noise solutions are uniformly attracted to zero in L^2 and in H^1. We will prove that because of noise, the behavior is completely different. Though for finite times the probabilities of large excursions off neighborhoods of zero go to zero exponentially fast with ϵ, if we wait long enough, the time scale is exponential, such large fluctuations occur and exit from a neighborhood of zero takes place. In the L^2 case, we only consider noises of additive type where, because of noise, mass is injected or pumped randomly in the system. It would also be possible to treat rather general multiplicative noises as long as noise allows injection of mass. In H^1 we consider the two types of noises.

We use large deviation techniques to prove the corresponding result in our infinite dimensional setting. In [68], the case of a space variable in a unidimensional torus is treated for a particular SPDE and the regularizing property of the Heat semi-group is a central tool. Let us stress that the Schrödinger linear group is an isometry on every Sobolev space based on L^2. In [29], the neighborhood is defined for a strong topology of β-Hölder functions and is relatively compact for a weaker topology, the space variable is again in a bounded subset of $\mathbb{R}^d$. Note also that one particular difficulty in infinite dimensions, along with compactness, is that the linear group is strongly and not uniformly continuous. In this article the neighborhood is not relatively compact, we work on the all space $\mathbb{R}^d$, the nonlinearity is locally Lipschitz only in H^1 for $d = 1$.

However, there remain difficult problems from the control of nonlinear PDEs to prove for example that the upper and lower bounds on the exit time are equal. Also, it seems formally that, in the case of a noise of additive type which is white in time and in space, the escape off levels of the Hamiltonian less than a constant is intimately related to the solitary waves. We will not address these last issues in the present article.

D.2 Preliminaries

Throughout the paper the following notations will be used.

The set of positive integers and positive real numbers are denoted by $\mathbb{N}^*$ and $\mathbb{R}_+^*$. For $p \in \mathbb{N}^*$, L^p is the Lebesgue space of complex valued functions. For k in $\mathbb{N}^*$, $\mathrm{W}^{k,p}$ is the Sobolev space of L^p functions with partial derivatives up to level k, in the sense of distributions, in L^p. For $p = 2$ and s in $\mathbb{R}_+^*$, H^s is the Sobolev space of tempered distributions v of Fourier transform $\hat{v}$ such that $(1 + |\xi|^2)^{s/2}\hat{v}$ belongs to L^2. We denote the spaces by $\mathrm{L}^p_{\mathbb{R}}$, $\mathrm{W}^{k,p}_{\mathbb{R}}$ and $\mathrm{H}^s_{\mathbb{R}}$ when the functions are real-valued. The space L^2 is endowed with the inner product $(u, v)_{\mathrm{L}^2} = \mathfrak{Re} \int_{\mathbb{R}^d} u(x)\overline{v}(x)dx$. If I is an interval of $\mathbb{R}$, $(E, \|\cdot\|_E)$ a Banach space and r belongs to $[1, \infty]$, then $\mathrm{L}^r(I; E)$ is the space of strongly Lebesgue measurable functions f from I into E such that $t \to \|f(t)\|_E$ is in $\mathrm{L}^r(I)$.

The space of linear continuous operators from B into $\tilde{B}$, where B and $\tilde{B}$ are Banach spaces is $\mathcal{L}_c\left(B, \tilde{B}\right)$. When $B = H$ and $\tilde{B} = \tilde{H}$ are Hilbert spaces, such an operator is Hilbert-Schmidt when $\sum_{j \in \mathbb{N}} \|\Phi e_j^H\|_{\tilde{H}}^2 < \infty$ for every $(e_j)_{j \in \mathbb{N}}$ complete orthonormal system of H. The set of such operators is denoted by $\mathcal{L}_2(H, \tilde{H})$, or $\mathcal{L}_2^{s,r}$ when $H = \mathrm{H}^s$ and $\tilde{H} = \mathrm{H}^r$. When $H = \mathrm{H}^s_{\mathbb{R}}$

and $\tilde{H} = \mathrm{H}^r_{\mathbb{R}}$, we denote it by $\mathcal{L}^{s,r}_{2,\mathbb{R}}$. When $s = 0$ or $r = 0$ the Hilbert space is L^2 or $\mathrm{L}^2_{\mathbb{R}}$.

We also denote by B^0_ρ and S^0_ρ respectively the open ball and the sphere centered at 0 of radius ρ in L^2. We denote these by B^1_ρ and S^1_ρ in H^1. We will denote by $\mathcal{N}^0(A,\rho)$ the $\rho-$neighborhood of a set A in L^2 and $\mathcal{N}^1(A,\rho)$ the neighborhood in H^1. In the following we impose that compact sets satisfy the Hausdorff property.

We will use in Lemma D.3.5 the integrability of the Schrödinger linear group which is related to the dispersive property. Recall that $(r(p),p)$ is an admissible pair if p is such that $2 \le p < \frac{2d}{d-2}$ when $d > 2$ ($2 \le p < \infty$ when $d = 2$ and $2 \le p \le \infty$ when $d = 1$) and $r(p)$ satisfies $\frac{2}{r(p)} = d\left(\frac{1}{2} - \frac{1}{p}\right)$.
For every $(r(p),p)$ admissible pair and T positive, we define the Banach spaces

$$Y^{(T,p)} = \mathrm{C}\left([0,T];\mathrm{L}^2\right) \cap \mathrm{L}^{r(p)}\left(0,T;\mathrm{L}^p\right),$$

and

$$X^{(T,p)} = \mathrm{C}\left([0,T];\mathrm{H}^1\right) \cap \mathrm{L}^{r(p)}\left(0,T;\mathrm{W}^{1,p}\right),$$

where the norms are the maximum of the norms in the two intersected Banach spaces. The Schrödinger linear group is denoted by $(U(t))_{t\ge 0}$; it is defined on L^2 or on H^1. Let us recall the Strichartz inequalities, see [25],

(i) There exists C positive such that for u_0 in L^2, T positive and $(r(p),p)$ admissible pair,

$$\|U(t)u_0\|_{Y^{(T,p)}} \le C\,\|u_0\|_{\mathrm{L}^2},$$

(ii) For every T positive, $(r(p),p)$ and $(r(q),q)$ admissible pairs, s and ρ such that $\frac{1}{s} + \frac{1}{r(q)} = 1$ and $\frac{1}{\rho} + \frac{1}{q} = 1$, there exists C positive such that for f in $\mathrm{L}^s(0,T;\mathrm{L}^\rho)$,

$$\left\|\int_0^{\cdot} U(\cdot - s)f(s)ds\right\|_{Y^{(T,p)}} \le C\|f\|_{\mathrm{L}^s(0,T;\mathrm{L}^\rho)}.$$

Similar inequalities hold when the group is acting on H^1, replacing L^2 by H^1, $Y^{(T,p)}$ by $X^{(T,p)}$ and $\mathrm{L}^s(0,T;\mathrm{L}^\rho)$ by $\mathrm{L}^s\left(0,T;\mathrm{W}^{1,\rho}\right)$.

It is known that, in the Hilbert space setting, only direct images of uncorrelated space wise Wiener processes by Hilbert-Schmidt operators are well defined. However, when the semi-group has regularizing properties, the semi-group may act as a Hilbert-Schmidt operator and a white in space noise may be considered. It is not possible here since the Schrödinger group is an isometry on the Sobolev spaces based on L^2. The Wiener process W

is thus defined as ΦW_c, where W_c is a cylindrical Wiener process on L^2 and Φ is Hilbert-Schmidt. Then $\Phi\Phi^*$ is the correlation operator of $W(1)$, it has finite trace.

We consider the following Cauchy problems

$$\begin{cases} i\mathrm{d}u^{\epsilon,u_0} &= (\Delta u^{\epsilon,u_0} + \lambda |u^{\epsilon,u_0}|^{2\sigma} u^{\epsilon,u_0} - i\alpha u^{\epsilon,u_0})\mathrm{d}t + \sqrt{\epsilon}\mathrm{d}W, \\ u^{\epsilon,u_0}(0) &= u_0 \end{cases} \tag{D.2.1}$$

with u_0 in L^2 and Φ in $\mathcal{L}_2^{0,0}$ or u_0 in H^1 and Φ in $\mathcal{L}_2^{0,1}$, and

$$\begin{cases} i\mathrm{d}u^{\epsilon,u_0} &= (\Delta u^{\epsilon,u_0} + \lambda |u^{\epsilon,u_0}|^{2\sigma} u^{\epsilon,u_0} - i\alpha u^{\epsilon,u_0})\mathrm{d}t + \sqrt{\epsilon} u^{\epsilon,u_0} \circ \mathrm{d}W, \\ u^{\epsilon,u_0}(0) &= u_0 \end{cases} \tag{D.2.2}$$

with u_0 in H^1 and Φ in $\mathcal{L}_{2,\mathbb{R}}^{0,s}$ where $s > \frac{d}{2} + 1$. When the noise is of multiplicative type, we may write the equation in terms of a Itô product,

$$i\mathrm{d}u^{\epsilon,u_0} = (\Delta u^{\epsilon,u_0} + \lambda |u^{\epsilon,u_0}|^{2\sigma} u^{\epsilon,u_0} - i\alpha u^{\epsilon,u_0} - \frac{i\epsilon}{2} u^{\epsilon,u_0} F_\Phi)\mathrm{d}t + \sqrt{\epsilon} u^{\epsilon,u_0}\mathrm{d}W,$$

where $F_\Phi(x) = \sum_{j\in\mathbb{N}} (\Phi e_j(x))^2$ for x in $\mathbb{R}^d$ and $(e_j)_{j\in\mathbb{N}}$ a complete orthonormal system of L^2. We consider mild solutions; for example the mild solutions of (D.2.1) satisfies

$$\begin{aligned} u^{\epsilon,u_0}(t) = \;& U(t)u_0 - i\lambda \int_0^t U(t-s)(|u^{\epsilon,u_0}(s)|^{2\sigma} u^{\epsilon,u_0}(s) - i\alpha u^{\epsilon,u_0}(s))ds \\ & -i\sqrt{\epsilon}\int_0^t U(t-s)dW(s), \quad t > 0. \end{aligned}$$

The Cauchy problems are globally well posed in L^2 and H^1 with the same arguments as in [37].

The main tools in this article are the sample paths LDPs for the solutions of the three Cauchy problems. They are uniform in the initial data. Unlike in [44, 81, 82] we use a Freidlin-Wentzell type formulation of the upper and lower bounds of the LDPs. Indeed it seems that the restriction that initial data be in compact sets in [82] is a real limitation in particular for stochastic NLS equations. Indeed the linear Schrödinger group is not compact due to the lack of smoothing effect and to the fact that we work on the whole space $\mathbb{R}^d$. This limitation disappears when we work with the Freidlin-Wentzell type formulation; we may now obtain bounds for initial data in balls of L^2 (respectively H^1) for ϵ small enough. Note that it is well known that in metric spaces and for non uniform LDPs the two formulations are equivalent. A proof will be given and we will stress, in the multiplicative case, on the slight differences with the proof of the result in [82].

We denote by $\mathbf{S}(u_0,h)$ the skeleton of equation (D.2.1) or (D.2.2), *i.e.* the mild solution of the controlled equation

$$\left\{\begin{array}{l} i\left(\frac{\mathrm{d}u}{\mathrm{d}t}+\alpha u\right)=\Delta u+\lambda|u|^{2\sigma}u+\Phi h,\\ u(0)=u_0\end{array}\right.$$

where u_0 belongs to L^2 or H^1 in the additive case and the mild solution of

$$\left\{\begin{array}{l} i\left(\frac{\mathrm{d}u}{\mathrm{d}t}+\alpha u\right)=\Delta u+\lambda|u|^{2\sigma}u+u\Phi h,\\ u(0)=u_0\end{array}\right.$$

where u_0 belongs to H^1 in the multiplicative case.
The rate functions of the LDPs are always defined as

$$I_T^{u_0}(w)=\frac{1}{2}\inf_{h\in\mathrm{L}^2(0,T;\mathrm{L}^2):\ \mathbf{S}(u_0,h)=w}\int_0^T\|h(s)\|_{\mathrm{L}^2}^2ds.$$

We denote for T and a positive by $K_T^{u_0}(a)=\left(I_T^{u_0}\right)^{-1}([0,a])$ the levels of the rate function less or equal to a

$$K_T^{u_0}(a)=\left\{w\in\mathrm{C}\left([0,T];\mathrm{L}^2\right):\ w=\mathbf{S}(u_0,h),\ \frac{1}{2}\int_0^T\|h(s)\|_{\mathrm{L}^2}^2ds\le a\right\}.$$

We also denote by $d_{\mathrm{C}([0,T];\mathrm{L}^2)}$ the usual distance between sets of $\mathrm{C}\left([0,T];\mathrm{L}^2\right)$ and by $d_{\mathrm{C}([0,T];\mathrm{H}^1)}$ the distance between sets of $\mathrm{C}\left([0,T];\mathrm{H}^1\right)$.

We also denote by $\tilde{\mathbf{S}}(u_0,f)$ the skeleton of equation (D.2.2) where we replace Φh by $\frac{\partial f}{\partial t}$ where f belongs to $\mathrm{H}_0^1\left(0,T;\mathrm{H}_{\mathbb{R}}^s\right)$, the subspace of $\mathrm{C}\left([0,T];\mathrm{H}_{\mathbb{R}}^s\right)$ of square integrable in time and with square integrable in time time derivative functions, null at $t=0$. Also C_a denotes the set

$$\left\{f\in\mathrm{H}_0^1\left(0,T;\mathrm{H}_{\mathbb{R}}^s\right):\ \frac{\partial f}{\partial t}\in\mathrm{Im}\Phi,\ I_T^W(f)=\frac{1}{2}\left\|\Phi^{-1}_{|(\mathrm{Ker}\Phi)^\perp}\frac{\partial f}{\partial t}\right\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)}\le a\right\}$$

and $\mathcal{A}(d)$ the set $[2,\infty)$ when $d=1$ or $d=2$ and $\left[2,\frac{2(3d-1)}{3(d-1)}\right)$ when $d\ge 3$. The above I_T^W is the good rate function of the LDP for the Wiener process. The uniform LDP with the Freidlin-Wentzell formulation that we will need in the remaining is then as follows. In the additive case we consider the L^2 and H^1 case while in the multiplicative case we only consider the H^1 case because we will not need a L^2 result. Indeed in the case of the multiplicative noise the L^2 norm remains invariant.

Theorem D.2.1 *In the additive case and in* L^2 *we have:*
for every a, ρ, T, δ *and* γ *positive,*

(i) *there exists* ϵ_0 *positive such that for every* ϵ *in* $(0,\epsilon_0)$, u_0 *such that* $\|u_0\|_{\mathrm{L}^2}\leq\rho$ *and* $\tilde{a}$ *in* $(0,a]$,

$$\mathbb{P}\left(d_{\mathrm{C}([0,T];\mathrm{L}^2)}\left(u^{\epsilon,u_0},K_T^{u_0}(\tilde{a})\right)\geq\delta\right)<\exp\left(-\frac{\tilde{a}-\gamma}{\epsilon}\right),$$

(ii) *there exists* ϵ_0 *positive such that for every* ϵ *in* $(0,\epsilon_0)$, u_0 *such that* $\|u_0\|_{\mathrm{L}^2}\leq\rho$ *and* w *in* $K_T^{u_0}(a)$,

$$\mathbb{P}\left(\|u^{\epsilon,u_0}-w\|_{\mathrm{C}([0,T];\mathrm{L}^2)}<\delta\right)>\exp\left(-\frac{I_T^{u_0}(w)+\gamma}{\epsilon}\right).$$

In H^1, *the result holds for additive and multiplicative noises replacing in the above* $\|u_0\|_{\mathrm{L}^2}$ *by* $\|u_0\|_{\mathrm{H}^1}$ *and* $\mathrm{C}\left([0,T];\mathrm{L}^2\right)$ *by* $\mathrm{C}\left([0,T];\mathrm{H}^1\right)$.

Note that the extra condition
For every a *positive and* K *compact in* L^2, *the set* $K_T^K(a)=\bigcup_{u_0\in K}K_T^{u_0}(a)$ *is a compact subset of* $\mathrm{C}\left([0,T];\mathrm{L}^2\right)$
often appears to be part of a uniform LDP. It will not be used in the following. The proof of this result is given in the annex.

D.3 Exit from a domain of attraction in L^2

In this section we only consider the case of an additive noise. Recall that for the real multiplicative noise the mass is decreasing and thus exit is impossible.

We may easily check that the mass $\mathbf{N}\left(\mathbf{S}(u_0,0)\right)$ of the solution of the deterministic equation satisfies

$$\mathbf{N}\left(\mathbf{S}(u_0,0)(t)\right)=\mathbf{N}\left(u_0\right)\exp\left(-2\alpha t\right). \tag{D.3.1}$$

With noise though, the mass fluctuates around the deterministic decay. Recall how the Itô formula applies to the fluctuation of the mass, see [37] for a proof,

$$\begin{array}{rl}\mathbf{N}\left(u^{\epsilon,u_0}(t)\right)-\mathbf{N}\left(u_0\right)= & -2\sqrt{\epsilon}\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t\overline{u}^{\epsilon,u_0}dW\,dx\\ & -2\alpha\left\|u^{\epsilon,u_0}\right\|^2_{\mathrm{L}^2(0,t;\mathrm{L}^2)}+\epsilon t\|\Phi\|^2_{\mathcal{L}_2^{0,0}}.\end{array} \tag{D.3.2}$$

Assume that D is a bounded measurable subset of L^2 containing 0 in its interior and invariant by the deterministic flow, *i.e.*

$$\forall u_0\in D,\ \forall t\geq0,\ \mathbf{S}(u_0,0)(t)\in D;$$

it may be an open ball. There exists R positive such that $D \subset B_R$.
We define by

$$\tau^{\epsilon,u_0} = \inf\left\{t \geq 0 :\ u^{\epsilon,u_0}(t) \in D^c\right\}$$

the first exit time of the process u^{ϵ,u_0} off the domain D.

An easy information on the exit time is obtained as follows. The expectation of an integration via the Duhamel formula of the Itô decomposition, the process u^{ϵ,u_0} being stopped at the first exit time, gives $\mathbb{E}\left[\exp\left(-2\alpha\tau^{\epsilon,u_0}\right)\right] = 1 - \frac{2\alpha R}{\epsilon\|\Phi\|^2_{\mathcal{L}_2^{0,0}}}$. Without damping we obtain $\mathbb{E}\left[\tau^{\epsilon,u_0}\right] = \frac{R}{\epsilon\|\Phi\|^2_{\mathcal{L}_2^{0,0}}}$. To get more precise information for small noises we use LDP techniques.

Let us introduce

$$\overline{e} = \inf\left\{I_T^0(w) :\ w(T) \in \overline{D}^c,\ T > 0\right\}.$$

When ρ is positive and small enough, we set

$$e_\rho = \inf\left\{I_T^{u_0}(w) : \|u_0\|_{\mathrm{L}^2} \leq \rho,\ w(T) \in \left(D_{-\rho}\right)^c,\ T > 0\right\},$$

where $D_{-\rho} = D \setminus \mathcal{N}^0\left(\partial D, \rho\right)$ and ∂D is the the boundary of ∂D in L^2. We define then

$$\underline{e} = \lim_{\rho \to 0} e_\rho.$$

We shall denote in this section by $\|\Phi\|_c$ the norm of Φ as a bounded operator on L^2. Let us start with the following lemma.

Lemma D.3.1 $0 < \underline{e} \leq \overline{e}$.

Proof. It is clear that $\underline{e} \leq \overline{e}$. Let us check that $\underline{e} > 0$. Let d denote the positive distance between 0 and ∂D. Take ρ small such that the distance between B_ρ^0 and $\left(D_{-\rho}\right)^c$ is larger than $\frac{d}{2}$. Multiplying the evolution equation by $-i\overline{\mathbf{S}(u_0,h)}$, taking the real part, integrating over space and using the Duhamel formula we obtain

$$\begin{aligned}
&\mathbf{N}\left(\mathbf{S}(u_0,h)(T)\right) - \exp\left(-2\alpha T\right)\mathbf{N}\left(u_0\right) \\
&= 2\int_0^T \exp\left(-2\alpha(T-s)\right)\mathfrak{Im}\left(\int_{\mathbb{R}^d}\overline{\mathbf{S}\left(u_0,h\right)}\Phi h dx ds\right).
\end{aligned}$$

If $\mathbf{S}(u_0,h)(T) \in \left(D_{-\rho}\right)^c$ and correspond to the first escape off D then

$$\begin{aligned}
\frac{d}{2} &\leq 2\|\Phi\|_c\int_0^T \exp\left(-2\alpha(T-s)\right)\|\mathbf{S}(u_0,h)(s)\|_{\mathrm{L}^2}\|h(s)\|_{\mathrm{L}^2} ds \\
&\leq 2R\|\Phi\|_c\left(\int_0^T \exp\left(-4\alpha(T-s)\right)ds\right)^{\frac{1}{2}}\|h\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)},
\end{aligned}$$

thus

$$\frac{\alpha d^2}{8R^2\|\Phi\|_c^2} \le \frac{1}{2}\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)},$$

and the result follows. □

Note that we would expect $\underline{e}$ and $\overline{e}$ to be equal. We should prove that, for a fixed level of energy, we may find ρ arbitrarily small and a control of energy less than the fixed level such that the controlled solution goes from 0 to u_0 in B_ρ^0 in finite time. We should also find a second control of energy smaller than the fixed level such that the controlled solution goes from $\partial D_{-\rho}$ to $\overline{D}^c$ in finite time. Note that control arguments for nonlinear Schrödinger equations where the control enters the equation as an external force or potential are used in [38, 39] in the study of the blow-up time for stochastic nonlinear Schrödinger equations. Here it seems more intricate and the arguments of [38, 39] do not seem to apply. If these two bounds were indeed equal, they would also correspond to

$$\begin{aligned}\mathcal{E}(D) = & \ \tfrac{1}{2}\inf\left\{\|h\|^2_{\mathrm{L}^2(0,\infty;\mathrm{L}^2)} : \ \exists T>0: \ \mathbf{S}(0,h)(T)\in\partial D\right\}\\ = & \ \inf_{v\in\partial D} V(0,v)\end{aligned}$$

where the quasi-potential is defined as

$$V(u_0,u_f) = \inf\left\{I_T^0(w): \ w\in \mathrm{C}\left(\mathbb{R}^+;\mathrm{L}^2\right), \ w(0)=u_0, \ w(T)=u_f, \ T>0\right\}.$$

We shall prove in this section the two following results. The first theorem characterizes the first exit time from the domain.

Theorem D.3.2 *For every u_0 in D and δ positive, there exists L positive such that*

$$\overline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(\tau^{\epsilon,u_0}\notin\left(\exp\left(\frac{\underline{e}-\delta}{\epsilon}\right),\exp\left(\frac{\overline{e}+\delta}{\epsilon}\right)\right)\right)\le -L, \tag{D.3.3}$$

and for every u_0 in D,

$$\underline{e}\le\underline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{E}\left(\tau^{\epsilon,u_0}\right)\le\overline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{E}\left(\tau^{\epsilon,u_0}\right)\le\overline{e}. \tag{D.3.4}$$

Moreover, for every δ positive, there exists L positive such that

$$\overline{\lim}_{\epsilon\to 0}\epsilon\log\sup_{u_0\in D}\mathbb{P}\left(\tau^{\epsilon,u_0}\ge\exp\left(\frac{\overline{e}+\delta}{\epsilon}\right)\right)\le -L, \tag{D.3.5}$$

and

$$\overline{\lim}_{\epsilon\to 0}\epsilon\log\sup_{u_0\in D}\mathbb{E}\left(\tau^{\epsilon,u_0}\right)\le\overline{e}. \tag{D.3.6}$$

The second theorem characterizes formally the exit points. We shall define for ρ positive small enough, N a closed subset of ∂D

$$e_{N,\rho} = \inf\left\{I_T^{u_0}(w) : \|u_0\|_{\mathrm{L}^2} \le \rho,\ w(T) \in \left(D \setminus \mathcal{N}^0(N,\rho)\right)^c,\ T > 0\right\}.$$

We then define

$$\underline{e}_N = \lim_{\rho\to 0} e_{N,\rho}.$$

Note that $e_\rho \le e_{N,\rho}$ and thus $\underline{e} \le \underline{e}_N$.

Theorem D.3.3 *If $\underline{e}_N > \overline{e}$, then for every u_0 in D, there exists L positive such that*

$$\overline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(u^{\epsilon,u_0}\left(\tau^{\epsilon,u_0}\right) \in N\right) \le -L.$$

Thus the probability of an escape off D via points of N such that $e_\rho \le e_{N,\rho}$ goes to zero exponentially fast with ϵ. Suppose that we were able to solve the previous control problem, then as noise goes to zero, the probability of an exit via closed subsets of ∂D where the quasi-potential is not minimal goes to zero. As the expected exit time is finite, an exit occurs almost surely. It is exponentially more likely that it occurs via infima of the quasi-potential. When there are several infima the exit measure is a probability measure on ∂D. When there exists only one infimum we may state the following corollary.

Corollary D.3.4 *Assume that v^* in ∂D is such that for every δ positive and $N = \{v \in \partial D :\ \|v - v^*\|_{\mathrm{L}^2} \ge \delta\}$ we have $\underline{e}_N > \overline{e}$ then*

$$\forall \delta > 0,\ \forall u_0 \in D,\ \exists L > 0 :\ \overline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(\left\|u^{\epsilon,u_0}\left(\tau^{\epsilon,u_0}\right) - v^*\right\|_{\mathrm{L}^2} \ge \delta\right) \le -L.$$

We need to prove a few lemmas before proving the two theorems.
Let us define

$$\sigma_\rho^{\epsilon,u_0} = \inf\left\{t \ge 0 : u^{\epsilon,u_0}(t) \in B_\rho^0 \cup D^c\right\},$$

where $B_\rho^0 \subset D$.

Lemma D.3.5 *For every ρ and L positive with $B_\rho^0 \subset D$, there exists T and ϵ_0 positive such that for every u_0 in D and ϵ in $(0,\epsilon_0)$,*

$$\mathbb{P}\left(\sigma_\rho^{\epsilon,u_0} > T\right) \le \exp\left(-\frac{L}{\epsilon}\right).$$

Proof. The result is straightforward if u_0 belongs to B^0_ρ. Suppose now that u_0 belongs to $D \setminus B^0_\rho$. From equation (D.3.1), the bounded subsets of L^2 are uniformly attracted to zero by the flow of the deterministic equation. Thus there exists a positive time T_1 such that for every u_1 in the $\frac{\rho}{8}-$neighborhood of $D \setminus B^0_\rho$ and $t \geq T_1$, $\mathbf{S}\left(u_1,0\right)(t) \in B^0_{\frac{\rho}{8}}$. We shall choose $\rho < 8$ and follow three steps.

Step 1: Let us first recall why there exists $M' = M'(T_1, R, \sigma, \alpha)$ such that

$$\sup_{u_1 \in \mathcal{N}^0\left(D \setminus B^0_\rho, \frac{\rho}{8}\right)} \|\mathbf{S}(u_1,0)\|_{Y^{(T_1,2\sigma+2)}} \leq M'. \tag{D.3.7}$$

From the Strichartz inequalities, there exists C positive such that

$$\begin{aligned} \|\mathbf{S}(u_1,0)\|_{Y^{(t,2\sigma+2)}} \leq \ & C\|u_1\|_{\mathrm{L}^2} + C\left\| |\mathbf{S}(u_1,0)|^{2\sigma+1}\right\|_{\mathrm{L}^{\gamma'}(0,t;\mathrm{L}^{s'})} \\ & + C\alpha \left\|\mathbf{S}\left(u_1,0\right)\right\|_{\mathrm{L}^1(0,t;\mathrm{L}^2)} \end{aligned}$$

where γ' and s' are such that $\frac{1}{\gamma'} + \frac{1}{r(\tilde{p})} = 1$ and $\frac{1}{s'} + \frac{1}{\tilde{p}} = 1$ and $(r(\tilde{p}), \tilde{p})$ is an admissible pair. Note that the first term is smaller than $C(R+1)$. From the Hölder inequality, setting

$$\frac{2\sigma}{2\sigma+2} + \frac{1}{2\sigma+2} = \frac{1}{s'}\ , \quad \frac{2\sigma}{\omega} + \frac{1}{r(2\sigma+2)} = \frac{1}{\gamma'},$$

we can write

$$\left\| |\mathbf{S}(u_1,0)|^{2\sigma+1}\right\|_{\mathrm{L}^{\gamma'}(0,t;\mathrm{L}^{s'})} \leq C\|\mathbf{S}(u_1,0)\|_{\mathrm{L}^{r(2\sigma+2)}(0,t;\mathrm{L}^{2\sigma+2})} \|\mathbf{S}(u_1,0)\|^{2\sigma}_{\mathrm{L}^{\omega}(0,t;\mathrm{L}^{2\sigma+2})}.$$

It is easy to check that since $\sigma < \frac{2}{d}$, we have $\omega < r(2\sigma+2)$. Thus it follows that

$$\|\mathbf{S}(u_1,0)\|_{Y^{(t,2\sigma+2)}} \leq C(R+1) + Ct^{\frac{\omega r(2\sigma+2)}{r(2\sigma+2)-\omega}} \|\mathbf{S}(u_1,0)\|^{2\sigma+1}_{Y^{(t,2\sigma+2)}} + C\alpha\sqrt{t}\,\|\mathbf{S}(u_1,0)\|_{Y^{(t,2\sigma+2)}}.$$

The function $x \mapsto C(R+1) + Ct^{\frac{\omega r(2\sigma+2)}{r(2\sigma+2)-\omega}} x^{2\sigma+1} + C\alpha\sqrt{t}x - x$ is positive on a neighborhood of zero. For $t_0 = t_0(R,\sigma,\alpha)$ small enough, the function has at least one zero. Also, the function goes to ∞ as x goes to ∞. Thus, denoting by $M(R,\sigma)$ the first zero of the above function, we obtain by a classical argument that $\|\mathbf{S}(u_1,0)\|_{Y^{(t_0,2\sigma+2)}} \leq M(R,\sigma)$ for every u_1 in $\mathcal{N}^0\left(D \setminus B^0_\rho, \frac{\rho}{8}\right)$. Also, as for every t in $[0,T]$, $\mathbf{S}(u_1,0)(t)$ belongs to $\mathcal{N}^0\left(D \setminus B^0_\rho, \frac{\rho}{8}\right)$, repeating the previous argument, u_1 is replaced by $\mathbf{S}(u_1,0)(t_0)$ and so on, we obtain

$$\sup_{u_1 \in \mathcal{N}^0\left(D \setminus B^0_\rho, \frac{\rho}{8}\right)} \|\mathbf{S}(u_1,0)\|_{Y^{(T_1,p)}} \leq M',$$

where $M' = \left\lceil \frac{T_1}{t_0} \right\rceil M$ proving (D.3.7).

Step2: Let us now prove that for T large enough, to be defined later, and larger than T_1, we have

$$\mathcal{T}_\rho = \left\{ w \in \mathrm{C}\left([0,T];\mathrm{L}^2\right) : \ \forall t \in [0,T], \ w(t) \in \mathcal{N}^0\left(D \setminus B^0_\rho, \frac{\rho}{8}\right)\right\} \subset K_T^{u_0}(2L)^c. \tag{D.3.8}$$

Since $K_T^{u_0}(2L)$ is included in the image of $\mathbf{S}(u_0,\cdot)$ it suffices to consider w in $\mathcal{T}_\rho$ such that $w = \mathbf{S}(u_0,h)$ for some h in $\mathrm{L}^2(0,T;\mathrm{L}^2)$. Take h such that $\mathbf{S}(u_0,h)$ belongs to $\mathcal{T}_\rho$ we have

$$\|\mathbf{S}(u_0,h) - \mathbf{S}(u_0,0)\|_{\mathrm{C}([0,T_1];\mathrm{L}^2)} \geq \|\mathbf{S}(u_0,h)(T_1) - \mathbf{S}(u_0,0)(T_1)\|_{\mathrm{L}^2} \geq \frac{3\rho}{4},$$

but also, necessarily, for the admissible pair $(r(2\sigma+2), 2\sigma+2)$,

$$\|\mathbf{S}(u_0,h) - \mathbf{S}(u_0,0)\|_{Y^{(T_1,2\sigma+2)}} \geq \frac{3\rho}{4}. \tag{D.3.9}$$

Denote by $\mathbf{S}^{M'+1}$ the skeleton corresponding to the following control problem

$$\begin{cases} i\left(\frac{\mathrm{d}u}{\mathrm{d}t} + \alpha u\right) = \Delta u + \lambda\theta\left(\frac{\|u\|_{Y^{(t,2\sigma+2)}}}{M'+1}\right)|u|^{2\sigma}u + \Phi h, \\ u(0) = u_1 \end{cases}$$

where θ is a C^∞ function with compact support, such that $\theta(x) = 0$ if $x \geq 2$ and $\theta(x) = 1$ if $0 \leq x \leq 1$. Then (D.3.9) implies that

$$\left\|\mathbf{S}^{M'+1}(u_0,h) - \mathbf{S}^{M'+1}(u_0,0)\right\|_{Y^{(T_1,2\sigma+2)}} \geq \frac{3\rho}{4}.$$

We shall now split the interval $[0,T_1]$ in many parts. We shall denote here by $Y^{s,t,2\sigma+2}$ for $s < t$ the space $Y^{t,2\sigma+2}$ on the interval $[s,t]$. Applying the Strichartz inequalities on a small interval $[0,t]$ with the computations in the proof of Lemma 3.3 in [36], we obtain

$$\begin{aligned} \left\|\mathbf{S}^{M'+1}(u_0,h) - \mathbf{S}^{M'+1}(u_0,0)\right\|_{Y^{(t,2\sigma+2)}} &\leq C\alpha\sqrt{t}\left\|\mathbf{S}^{M'+1}(u_0,h) - \mathbf{S}^{M'+1}(u_0,0)\right\|_{Y^{(t,2\sigma+2)}} \\ &+ C_{M'+1}t^{1-\frac{d\sigma}{2}}\left\|\mathbf{S}^{M'+1}(u_0,h) - \mathbf{S}^{M'+1}(u_0,0)\right\|_{Y^{(t,2\sigma+2)}} + C\sqrt{t}\|\Phi\|_c\|h\|_{\mathrm{L}^2(0,t;\mathrm{L}^2)} \end{aligned}$$

where $C_{M'+1}$ is a constant which depends on $M'+1$. Take t_1 small enough such that $C_{M'+1}t_1^{1-\frac{\sigma d}{2}} + C\alpha\sqrt{t_1} \leq \frac{1}{2}$. We obtain then

$$\left\|\mathbf{S}^{M'+1}(u_0,h) - \mathbf{S}^{M'+1}(u_0,0)\right\|_{Y^{(t_1,2\sigma+2)}} \leq 2C\sqrt{t_1}\|\Phi\|_c\|h\|_{\mathrm{L}^2(0,t_1;\mathrm{L}^2)}.$$

In the case where $2t_1 < T_1$, let us see how such inequality propagates on $[t_1, 2t_1]$. We now have two different initial data $\mathbf{S}^{M'+1}(u_0, h)\,(t_1)$ and $\mathbf{S}^{M'+1}(u_0, 0)\,(t_1)$. We obtain similarly

$$\begin{aligned}
&\left\|\mathbf{S}^{M'+1}(u_0,h) - \mathbf{S}^{M'+1}(u_0,0)\right\|_{Y(t_1,2t_1,2\sigma+2)} \\
&\leq 2C\sqrt{t_1}\|\Phi\|_c\|h\|_{\mathrm{L}^2(0,t_1;\mathrm{L}^2)} + 2\left\|\mathbf{S}^{M'+1}(u_0,h)\,(t_1) - \mathbf{S}^{M'+1}(u_0,0)\,(t_1)\right\|_{\mathrm{H}^1} \\
&\leq 2C\sqrt{t_1}\|\Phi\|_c\|h\|_{\mathrm{L}^2(0,T_1;\mathrm{L}^2)} + 2\left\|\mathbf{S}^{M'+1}(u_0,h)\,(t_1) - \mathbf{S}^{M'+1}(u_0,0)\,(t_1)\right\|_{Y(0,t_1,2\sigma+2)}.
\end{aligned}$$

Then iterating on each interval of the form $[kt_1, (k+1)t_1]$ for k in $\left\{1, ..., \left\lfloor \frac{T_1}{t_1} - 1\right\rfloor\right\}$, the remaining term can be treated similarly, and using the triangle inequality we obtain that

$$\left\|\mathbf{S}^{M'+1}(u_0,h) - \mathbf{S}^{M'+1}(u_0,0)\right\|_{Y(T_1,2\sigma+2)} \leq 2^{\left\lceil \frac{T_1}{t_1}\right\rceil+1} C\sqrt{t_1}\|\Phi\|_c\|h\|_{\mathrm{L}^2(0,t_1;\mathrm{L}^2)}.$$

We may then conclude that

$$\frac{1}{2}\|h\|^2_{\mathrm{L}^2(0,T_1;\mathrm{L}^2)} \geq M''$$

where $M'' = \frac{\rho^2}{8C(t_1,T_1)\|\Phi\|_c^2}$ and $C\,(t_1, T_1)$ is a constant which depends only on t_1 and T_1. Note that we have used for later purposes that $\frac{3\rho}{2} > \frac{\rho}{2}$.

Similarly replacing $[0, T_1]$ by $[T_1, 2T_1]$ and u_0 respectively by $\mathbf{S}\,(u_0, h)\,(T_1)$ and $\mathbf{S}\,(u_0, 0)\,(T_1)$ in (D.3.9), the inequality still holds true. Thus thanks to the inverse triangle inequality we obtain on $[T_1, 2T_1]$

$$\begin{aligned}
&\left\|\mathbf{S}^{M'+1}(u_0,h) - \mathbf{S}^{M'+1}(u_0,0)\right\|_{Y(T_1,2T_1,2\sigma+2)} \\
&= \left\|\mathbf{S}^{M'+1}\left(\mathbf{S}^{M'+1}(u_0,h)(T_1), h\right) - \mathbf{S}^{M'+1}\left(\mathbf{S}^{M'+1}(u_0,0)(T_1), 0\right)\right\|_{Y(0,T_1,2\sigma+2)} \\
&\geq \tfrac{3\rho}{4}
\end{aligned}$$

Thus from the inverse triangle inequality along with the fact that for both $\mathbf{S}^{M'+1}(u_0,h)(T_1)$ and $\mathbf{S}^{M'+1}(u_0,0)(T_1)$ as initial data the deterministic solutions belong to the ball $B^0_{\frac{\rho}{8}}$, we obtain

$$\left\|\mathbf{S}^{M'+1}\left(\mathbf{S}^{M'+1}(u_0,h)(T_1), h\right) - \mathbf{S}^{M'+1}\left(\mathbf{S}^{M'+1}(u_0,h)(T_1), 0\right)\right\|_{Y(0,T_1,2\sigma+2)} \geq \frac{\rho}{2}.$$

We finally obtain the same lower bound

$$\frac{1}{2}\|h\|^2_{\mathrm{L}^2(T_1,2T_1;\mathrm{L}^2)} \geq M''$$

as before.
Iterating the argument we obtain if $T > 2T_1$,

$$\frac{1}{2}\|h\|^2_{L^2(0,2T_1;L^2)} = \frac{1}{2}\|h\|^2_{L^2(0,T_1;L^2)} + \frac{1}{2}\|h\|^2_{L^2(T_1,2T_1;L^2)} \geq 2M''.$$

Thus for j positive and $T > jT_1$, we obtain, iterating the above argument, that

$$\frac{1}{2}\|h\|^2_{L^2(0,jT_1;L^2)} \geq jM''.$$

The result (D.3.8) is obtained for $T = jT_1$ where j is such that $jM'' > 2L$.

Step 3: We may now conclude from the (i) of Theorem D.2.1 since,

$$\begin{aligned}\mathbb{P}\left(\sigma_\rho^{\epsilon,u_0} > T\right) &= \mathbb{P}\left(\forall t \in [0,T],\ u^{\epsilon,u_0}(t) \in D \setminus B_\rho^0\right)\\ &= \mathbb{P}\left(d_{C([0,T];L^2)}\left(u^{\epsilon,u_0}, \mathcal{T}_\rho^c\right) > \tfrac{\rho}{8}\right),\\ &\leq \mathbb{P}\left(d_{C([0,T];L^2)}\left(u^{\epsilon,u_0}, K_T^{u_0}(2L)\right) \geq \tfrac{\rho}{8}\right),\end{aligned}$$

taking $a = 2L$, $\rho = R$ where $D \subset B_R$, $\delta = \frac{\rho}{8}$ and $\gamma = L$.

Note that if $\rho \geq 8$, we should replace $R + 1$ by $R + \frac{\rho}{8}$ and $M' + 1$ by $M' + \frac{\rho}{8}$. Anyway, we will use the lemma for small ρ. $\square$

Lemma D.3.6 *For every ρ positive such that $B_\rho^0 \subset D$ and u_0 in D, there exists L positive such that*

$$\overline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(u^{\epsilon,u_0}\left(\sigma_\rho^{\epsilon,u_0}\right) \in \partial D\right) \leq -L$$

Proof. Take ρ positive satisfying the assumptions of the lemma and take u_0 in D. When u_0 belongs to B_ρ^0 the result is straightforward. Suppose now that u_0 belongs to $D \setminus B_\rho^0$. Let T be defined as

$$T = \inf\left\{t \geq 0 :\ \mathbf{S}(u_0,0)(t) \in B^0_{\frac{\rho}{2}}\right\},$$

then since $\mathbf{S}(u_0,0)([0,T])$ is a compact subset of D, the distance d between $\mathbf{S}(u_0,0)([0,T])$ and D^c is well defined and positive. The conclusion follows then from the fact that

$$\mathbb{P}\left(u^{\epsilon,u_0}\left(\sigma_\rho^{\epsilon,u_0}\right) \in \partial D\right) \leq \mathbb{P}\left(\|u^{\epsilon,u_0} - \mathbf{S}(u_0,0)\|_{C([0,T];L^2)} \geq \frac{\rho \wedge d}{2}\right),$$

the LDP and the fact that, from the compactness of the sets $K_T^{u_0}(a)$ for a positive, we have

$$\inf_{h \in L^2(0,T;L^2):\ \|\mathbf{S}(u_0,h) - \mathbf{S}(u_0,0)\|_{C([0,T];L^2)} \geq \frac{\rho\wedge d}{2}} \|h\|^2_{L^2(0,T;L^2)} > 0.$$

We have used the fact that the upper bound of the LDP in the Freidlin-Wentzell formulation implies the classical upper bound. Note that this is a well known result for non uniform LDPs. Indeed we do not need a uniform LDP in this proof. □

The following lemma replaces Lemma 5.7.23 in [48]. Indeed, the case of a stochastic PDE is more intricate than that of a SDE since the linear group is only strongly and not uniformly continuous. However, it is possible to prove that the group on L^2 when acting on bounded sets of H^1 is uniformly continuous. We shall proceed in a different manner and thus we will not loose in regularity. Indeed, the Schrödinger group does not have regularizing properties and we would obtain a weaker result with extra assumptions on Φ and the initial data.

Lemma D.3.7 *For every ρ and L positive such that $B^0_{2\rho} \subset D$, there exists $T(L,\rho) < \infty$ such that*

$$\overline{\lim}_{\epsilon\to 0}\epsilon \log \sup_{u_0\in S^0_\rho} \mathbb{P}\left(\sup_{t\in[0,T(L,\rho)]} \left(\mathbf{N}\left(u^{\epsilon,u_0}(t)\right) - \mathbf{N}\left(u_0\right)\right) \geq 3\rho^2\right) \leq -L$$

Proof. Take L and ρ positive. Note that for every ϵ in $(0,\epsilon_0)$ where $\epsilon_0 = \frac{\rho^2}{\|\Phi\|^2_{\mathcal{L}_2^{0,0}}}$, for $T(L,\rho) \leq 1$ we have $\epsilon T(L,\rho)\|\Phi\|^2_{\mathcal{L}_2^{0,0}} < \rho^2$. Thus from equation (D.3.2), we know that it is enough to prove that there exists $T(L,\rho) \leq 1$ such that for ϵ_1 small enough, $\epsilon_1 < \epsilon_0$, and all $\epsilon < \epsilon_0$,

$$\epsilon \log \sup_{u_0\in S^0_\rho} \mathbb{P}\left(\sup_{t\in[0,T(L,\rho)]} \left(-2\sqrt{\epsilon}\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t \overline{u}^{\epsilon,u_0,\tau} dW dx\right) \geq 2\rho^2\right) \leq -L,$$

where $u^{\epsilon,u_0,\tau}$ is the process u^{ϵ,u_0} stopped at $\tau^{\epsilon,u_0}_{S^0_{2\rho}}$, the first time when u^{ϵ,u_0} hits $S^0_{2\rho}$. Setting $Z(t) = \mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t \overline{u}^{\epsilon,u_0,\tau} dW dx$, it is enough to show that

$$\epsilon \log \sup_{u_0\in S^0_\rho} \mathbb{P}\left(\sup_{t\in[0,T(L,\rho)]} |Z(t)| \geq \frac{\rho^2}{\sqrt{\epsilon}}\right) \leq -L,$$

and thus to show exponential tail estimates for the process $Z(t)$. Our proof now follows closely that of [117][Theorem 2.1]. We introduce the function $f_l(x) = \sqrt{1+lx^2}$, where l is a positive parameter. We now apply the Itô formula to $f_l(Z(t))$ and the process decomposes into $1 + E_l(t) + R_l(t)$ where

$$E_l(t) = \int_0^t \frac{2lZ(t)}{\sqrt{1+lZ(t)^2}} dZ(t) - \frac{1}{2}\int_0^t \left(\frac{2lZ(t)}{\sqrt{1+lZ(t)^2}}\right)^2 d < Z >_t,$$

and

$$R_l(t) = \frac{1}{2}\int_0^t \left(\frac{2lZ(t)}{\sqrt{1+lZ(t)^2}}\right)^2 d < Z >_t + \int_0^t \frac{l}{(1+lZ(t)^2)^{\frac{3}{2}}} d < Z >_t .$$

Moreover, given $(e_j)_{j\in\mathbb{N}}$ a complete orthonormal system of L^2,

$$< Z(t) >= \int_0^t \sum_{j\in\mathbb{N}} \left(u^{\epsilon,u_0,\tau}, -i\Phi e_j\right)_{\mathrm{L}^2}^2 (s)ds,$$

we prove with the Hölder inequality that $|R_l(t)| \leq 12l\rho^2\|\Phi\|^2_{\mathcal{L}_2^{0,0}}t$, for every u_0 in D. We may thus write

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\in[0,T(L,\rho)]}|Z(t)| \geq \tfrac{\rho^2}{\sqrt{\epsilon}}\right)\\
&= \mathbb{P}\left(\sup_{t\in[0,T(L,\rho)]}\exp\left(f_l(Z(t))\right) \geq \exp\left(f_l\left(\tfrac{\rho^2}{\sqrt{\epsilon}}\right)\right)\right)\\
&\leq \mathbb{P}\left(\sup_{t\in[0,T(L,\rho)]}\exp\left(E_l(t)\right) \geq \exp\left(f_l\left(\tfrac{\rho^2}{\sqrt{\epsilon}}\right) - 1 - 12l\rho^2\|\Phi\|^2_{\mathcal{L}_2^{0,0}}T(L,\rho)\right)\right).
\end{aligned}$$

The Novikov condition is also satisfied and $E_l(t)$ is such that $(\exp(E_l(t)))_{t\in\mathbb{R}^+}$ is a uniformly integrable martingale. The exponential tail estimates follow from the Doob inequality optimizing on the parameter l. We may then write

$$\sup_{u_0\in S^0_\rho}\mathbb{P}\left(\sup_{t\in[0,T(L,\rho)]}|Z(t)| \geq \frac{\rho^2}{\sqrt{\epsilon}}\right) \leq 3\exp\left(-\frac{\rho^2}{48\epsilon\|\Phi\|^2_{\mathcal{L}_2^{0,0}}T(L,\rho)}\right).$$

We now conclude setting $T(L,\rho) = \frac{\rho^2}{50\|\Phi\|^2_{\mathcal{L}_2^{0,0}}L}$ and choosing $\epsilon_1 < \epsilon_0$ small enough. □

Proof of Theorem D.3.2. Let us first prove (D.3.6) and deduce (D.3.5). Fix δ positive and choose h and T_1 such that $\mathbf{S}(0,h)(T_1) \in \overline{D}^c$ and

$$I^0_{T_1}\left(\mathbf{S}(0,h)\right) = \frac{1}{2}\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \leq \overline{e} + \frac{\delta}{5}.$$

Let d_0 denote the positive distance between $\mathbf{S}(0,h)\left(T_1\right)$ and $\overline{D}$. With similar arguments as in [37] or with a truncation argument we may prove that the skeleton is continuous with respect to the initial datum for the L^2 topology. Thus there exists ρ positive, a function of h which has been fixed, such that if u_0 belongs to B^0_ρ then

$$\|\mathbf{S}\left(u_0,h\right) - \mathbf{S}(0,h)\|_{\mathrm{C}([0,T_1];\mathrm{L}^2)} < \frac{d_0}{2}.$$

We may assume that ρ is such that $B_\rho^0 \subset D$. From the triangle inequality and the (ii) of Theorem D.2.1, there exists ϵ_1 positive such that for all ϵ in $(0,\epsilon_1)$ and u_0 in B_ρ^0,

$$\begin{aligned}\mathbb{P}\left(\tau^{\epsilon,u_0} < T_1\right) &\geq \mathbb{P}\left(\|u^{\epsilon,u_0} - \mathbf{S}(0,h)\|_{\mathrm{C}([0,T_1];\mathrm{L}^2)} < d_0\right)\\ &\geq \mathbb{P}\left(\|u^{\epsilon,u_0} - \mathbf{S}\left(u_0,h\right)\|_{\mathrm{C}([0,T_1];\mathrm{L}^2)} < \frac{d_0}{2}\right)\\ &\geq \exp\left(-\frac{I_{T_1}^{u_0}(\mathbf{S}(u_0,h))+\frac{\delta}{5}}{\epsilon}\right).\end{aligned}$$

From Lemma D.3.5, there exists T_2 and ϵ_2 positive such that for all ϵ in $(0,\epsilon_2)$,

$$\inf_{u_0\in D}\mathbb{P}\left(\sigma_\rho^{\epsilon,u_0} \leq T_2\right) \geq \frac{1}{2}.$$

Thus, for $T = T_1 + T_2$, from the strong Markov property we obtain that for all $\epsilon < \epsilon_3 < \epsilon_1 \wedge \epsilon_2$.

$$\begin{aligned}q = \inf_{u_0\in D}\mathbb{P}\left(\tau^{\epsilon,u_0} \leq T\right) &\geq \inf_{u_0\in D}\mathbb{P}\left(\sigma_\rho^{\epsilon,u_0} \leq T_2\right)\inf_{u_0\in B_\rho^0}\mathbb{P}\left(\tau^{\epsilon,u_0} \leq T_1\right)\\ &\geq \frac{1}{2}\exp\left(-\frac{I_{T_1}^{u_0}(\mathbf{S}(u_0,h))+\frac{\delta}{5}}{\epsilon}\right)\\ &\geq \exp\left(-\frac{I_{T_1}^{u_0}(\mathbf{S}(u_0,h))+\frac{2\delta}{5}}{\epsilon}\right).\end{aligned}$$

Thus, for any $k \geq 1$, we have

$$\begin{aligned}\mathbb{P}\left(\tau^{\epsilon,u_0} > (k+1)T\right) &= \left[1 - \mathbb{P}\left(\tau^{\epsilon,u_0} \leq (k+1)T \middle| \tau^{\epsilon,u_0} > kT\right)\right]\mathbb{P}\left(\tau^{\epsilon,u_0} > kT\right)\\ &\leq (1-q)\mathbb{P}\left(\tau^{\epsilon,u_0} > kT\right)\\ &\leq (1-q)^k.\end{aligned}$$

We may now compute, since $I_{T_1}^{u_0}\left(\mathbf{S}\left(u_0,h\right)\right) = I_{T_1}^{0}\left(\mathbf{S}\left(0,h\right)\right) = \frac{1}{2}\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)}$

$$\begin{aligned}\sup_{u_0\in D}\mathbb{E}\left(\tau^{\epsilon,u_0}\right) &= \sup_{u_0\in D}\int_0^\infty \mathbb{P}\left(\tau^{\epsilon,u_0} > t\right)dt\\ &\leq T\left[1 + \sum_{k=1}^\infty \sup_{x\in D}\mathbb{P}\left(\tau^{\epsilon,u_0} > kT\right)\right]\\ &\leq \frac{T}{q}\\ &\leq T\exp\left(\frac{\overline{e}+\frac{3\delta}{5}}{\epsilon}\right).\end{aligned}$$

It implies that there exists ϵ_4 small enough such that for ϵ in $(0,\epsilon_4)$,

$$\sup_{u_0\in D}\mathbb{E}\left(\tau^{\epsilon,u_0}\right) \leq \exp\left(\frac{\overline{e}+\frac{4\delta}{5}}{\epsilon}\right). \tag{D.3.10}$$

Thus the Chebychev inequality gives that

$$\sup_{u_0\in D}\mathbb{P}\left(\tau^{\epsilon,u_0}\geq \exp\left(\frac{\overline{e}+\delta}{\epsilon}\right)\right)\leq \exp\left(-\frac{\overline{e}+\delta}{\epsilon}\right)\sup_{u_0\in D}\mathbb{E}\left(\tau^{\epsilon,u_0}\right),$$

in other words

$$\sup_{u_0\in D}\mathbb{P}\left(\tau^{\epsilon,u_0}\geq \exp\left(\frac{\overline{e}+\delta}{\epsilon}\right)\right)\leq \exp\left(-\frac{\delta}{5\epsilon}\right). \tag{D.3.11}$$

Relations (D.3.10) and (D.3.11) imply (D.3.6) and (D.3.5).

Let us now prove the lower bound on τ^{ϵ,u_0}. Take δ positive. Remind that we have proved that $\underline{e}>0$. Take ρ positive small enough such that $\underline{e}-\frac{\delta}{4}\leq e_\rho$ and $B^0_{2\rho}\subset D$. We define the following sequences of stopping times, $\theta_0=0$ and for k in $\mathbb{N}$,

$$\begin{array}{rl}\tau_k & =\inf\left\{t\geq\theta_k:\ u^{\epsilon,u_0}(t)\in B^0_\rho\cup D^c\right\},\\ \theta_{k+1} & =\inf\left\{t>\tau_k:\ u^{\epsilon,u_0}(t)\in S^0_{2\rho}\right\},\end{array}$$

where $\theta_{k+1}=\infty$ if $u^{\epsilon,u_0}(\tau_k)\in\partial D$. Fix $T_1=T\left(\underline{e}-\frac{3\delta}{4},\rho\right)$ given in Lemma D.3.7. We know that there exists ϵ_1 positive such that for all ϵ in $(0,\epsilon_1)$, for all $k\geq 1$ and u_0 in D,

$$\mathbb{P}\left(\theta_k-\tau_{k-1}\leq T_1\right)\leq \exp\left(-\frac{\underline{e}-\frac{3\delta}{4}}{\epsilon}\right).$$

For u_0 in D and an m in $\mathbb{N}^*$, we have

$$\begin{array}{rl}\mathbb{P}\left(\tau^{\epsilon,u_0}\leq mT_1\right)\leq & \mathbb{P}\left(\tau^{\epsilon,u_0}=\tau_0\right)+\sum_{k=1}^m\mathbb{P}\left(\tau^{\epsilon,u_0}=\tau_k\right)\\ & +\mathbb{P}\left(\exists k\in\{1,...,m\}:\ \theta_k-\tau_{k-1}\leq T_1\right)\\ = & \mathbb{P}\left(\tau^{\epsilon,u_0}=\tau_0\right)+\sum_{k=1}^m\mathbb{P}\left(\tau^{\epsilon,u_0}=\tau_k\right)\\ & +\sum_{k=1}^m\mathbb{P}\left(\theta_k-\tau_{k-1}\leq T_1\right).\end{array} \tag{D.3.12}$$

In other words the escape before mT_1 can occur either as an escape without passing in the small ball B^0_ρ (if u_0 belongs to $D\setminus B^0_\rho$) or as an escape with k in $\{1,...m\}$ significant fluctuations off B^0_ρ, *i.e.* crossing $S^0_{2\rho}$, or at least one of the m first transitions between S^0_ρ and $S^0_{2\rho}$ happens in less than T_1. The latter is known to be arbitrarily small. Let us prove that the remaining probabilities are small enough for small ϵ.
For every $k\geq 1$ and T_2 positive, we may write

$$\mathbb{P}\left(\tau^{\epsilon,u_0}=\tau_k\right)\leq\mathbb{P}\left(\tau^{\epsilon,u_0}\leq T_2;\tau^{\epsilon,u_0}=\tau_k\right)+\mathbb{P}\left(\sigma_\rho^{\epsilon,u_0}>T_2\right).$$

Fix T_2 as in Lemma D.3.5 with $L = \underline{e} - \frac{3\delta}{4}$. Thus there exists ϵ_2 small enough such that for ϵ in $(0, \epsilon_2)$,

$$\mathbb{P}\left(\sigma_\rho^{\epsilon,u_0} > T_2\right) \leq \exp\left(-\frac{\underline{e} - \frac{3\delta}{4}}{\epsilon}\right).$$

Also, from the (i) of Theorem D.2.1, we obtain that there exists ϵ_3 positive such that for every u_1 in B_ρ^0 and ϵ in $(0, \epsilon_3)$,

$$\begin{aligned}\mathbb{P}\left(\tau^{\epsilon,u_1} \leq T_2\right) &\leq \mathbb{P}\left(d_{\mathrm{C}([0,T_2];\mathrm{L}^2)}\left(u^{\epsilon,u_1}, K_{T_2}^{u_1}\left(e_\rho - \tfrac{\delta}{4}\right)\right) \geq \rho\right)\\ &\leq \exp\left(-\tfrac{e_\rho - \frac{\delta}{2}}{\epsilon}\right)\\ &\leq \exp\left(-\tfrac{\underline{e} - \frac{3\delta}{4}}{\epsilon}\right).\end{aligned}$$

Thus the above bound holds for $\mathbb{P}\left(\tau^{\epsilon,u_0} \leq T_2; \tau^{\epsilon,u_0} = \tau_k\right)$ replacing u_1 by $u^{\epsilon,u_0}\left(\tau_{k-1}\right)$ since as $k \geq 1$, $u^{\epsilon,u_0}\left(\tau_{k-1}\right)$ belongs to B_ρ^0 and $\tau_k - \tau_{k-1} \leq T_2$ and using the Markov property. The inequality (D.3.12) gives that for all ϵ in $(0, \epsilon_0)$ where $\epsilon_0 = \epsilon_1 \wedge \epsilon_2 \wedge \epsilon_3$,

$$\mathbb{P}\left(\tau^{\epsilon,u_0} \leq mT_1\right) \leq \mathbb{P}\left(u^{\epsilon,u_0}\left(\sigma_\rho^{\epsilon,u_0}\right) \in \partial D\right) + 3m \exp\left(-\frac{\underline{e} - \frac{3\delta}{4}}{\epsilon}\right).$$

Fix $m = \left\lceil \frac{1}{T_1} \exp\left(\frac{\underline{e} - \delta}{\epsilon}\right)\right\rceil$, then for all ϵ in $(0, \epsilon_0)$,

$$\begin{aligned}\mathbb{P}\left(\tau^{\epsilon,u_0} \leq \exp\left(\tfrac{\underline{e} - \delta}{\epsilon}\right)\right) &\leq \mathbb{P}\left(\tau^{\epsilon,u_0} \leq mT_1\right)\\ &\leq \mathbb{P}\left(u^{\epsilon,u_0}\left(\sigma_\rho^{\epsilon,u_0}\right) \in \partial D\right) + \tfrac{3}{T_1} \exp\left(-\tfrac{\delta}{4\epsilon}\right).\end{aligned}$$

We may now conclude with Lemma D.3.6 and obtain the expected lower bound on $\mathbb{E}\left(\tau^{\epsilon,u_0}\right)$ from the Chebychev inequality. $\square$

Proof of Theorem D.3.3. Let N be closed subset of ∂D. When $\underline{e}_N = \infty$ we shall replace in the proof that follows $\underline{e}_N$ by an increasing sequence of positive numbers. Take δ such that $0 < \delta < \frac{\underline{e}_N - \overline{e}}{3}$, ρ positive such that $\underline{e}_N - \frac{\delta}{3} \leq e_{N,\rho}$ and $B_{2\rho}^0 \subset D$. Define the same sequences of stopping times $\left(\tau_k\right)_{k \in \mathbb{N}}$ and $\left(\theta_k\right)_{k \in \mathbb{N}}$ as in the proof of Theorem D.3.2.
Take $L = \underline{e}_N - \delta$ and T_1 and $T_2 = T(L, \rho)$ as in Lemma D.3.5 and D.3.7. Thanks to Lemma D.3.5 and the uniform LDP, with a computation similar to the one following inequality (D.3.12), we obtain that for ϵ_0 small enough

and $\epsilon \leq \epsilon_0$,

$$\begin{aligned}
&\sup_{u_0 \in S^0_{2\rho}} \mathbb{P}\left(u^{\epsilon,u_0}\left(\sigma_\rho^{\epsilon,u_0}\right) \in N\right)\\
&\leq \sup_{u_0 \in S^0_{2\rho}} \mathbb{P}\left(u^{\epsilon,u_0}\left(\sigma_\rho^{\epsilon,u_0}\right) \in N, \sigma_\rho^{\epsilon,u_0} \leq T_1\right) + \sup_{u_0 \in S^0_{2\rho}} \mathbb{P}\left(\sigma_\rho^{\epsilon,u_0} > T_1\right)\\
&\leq \sup_{u_0 \in B^0_{2\rho}} \mathbb{P}\left(d_{\mathrm{C}([0,T_1];\mathrm{L}^2)}\left(u^{\epsilon,u_0}, K_{T_1}^{u_0}\left(e_{N,\rho} - \frac{\delta}{3}\right)\right) \geq \rho\right)\\
&\quad + \sup_{u_0 \in D} \mathbb{P}\left(\sigma_\rho^{\epsilon,u_0} > T_1\right)\\
&\leq 2\exp\left(-\frac{\underline{e}_N - \delta}{\epsilon}\right).
\end{aligned}$$

Possibly choosing ϵ_0 smaller, we may assume that for every positive integer l and every $\epsilon \leq \epsilon_0$,

$$\begin{aligned}
\sup_{u_0 \in D} \mathbb{P}\left(\tau_l \leq lT_2\right) \leq\ & l \sup_{u_0 \in S^0_\rho} \mathbb{P}\left(\sup_{t \in [0,T_2]}\left(\mathbf{N}\left(u^{\epsilon,u_0}(t)\right) - \mathbf{N}\left(u_0\right)\right) \geq \rho\right)\\
\leq\ & l \exp\left(-\frac{\underline{e}_N - \delta}{\epsilon}\right).
\end{aligned}$$

Thus if u_0 belongs to B^0_ρ

$$\begin{aligned}
\mathbb{P}\left(u^{\epsilon,u_0}\left(\tau^{\epsilon,u_0}\right) \in N\right) \leq\ & \mathbb{P}\left(\tau^{\epsilon,u_0} > \tau_l\right) + \sum_{k=1}^l \mathbb{P}\left(u^{\epsilon,u_0}\left(\tau^{\epsilon,u_0}\right) \in N, \tau^{\epsilon,u_0} = \tau_k\right)\\
\leq\ & \mathbb{P}\left(\tau^{\epsilon,u_0} > lT_2\right) + \mathbb{P}\left(\tau_l \leq lT_2\right)\\
& + l \sup_{u_0 \in S^0_{2\rho}} \mathbb{P}\left(u^{\epsilon,u_0}\left(\sigma_\rho^{\epsilon,u_0}\right) \in N\right)\\
\leq\ & \mathbb{P}\left(\tau^{\epsilon,u_0} > lT_2\right) + 3l \exp\left(-\frac{\underline{e}_N - \delta}{\epsilon}\right).
\end{aligned}$$

Take now $l = \left\lceil \frac{1}{T_2} \exp\left(\frac{\overline{e} + \delta}{\epsilon}\right)\right\rceil$ and use the upper bound (D.3.11), possibly choosing ϵ_0 smaller, we obtain that for $\epsilon \leq \epsilon_0$

$$\begin{aligned}
\sup_{u_0 \in B^0_\rho} \mathbb{P}\left(u^{\epsilon,u_0}\left(\tau^{\epsilon,u_0}\right) \in N\right) \leq\ & \exp\left(-\frac{\delta}{5\epsilon}\right) + \frac{4}{T_2} \exp\left(-\frac{\underline{e}_N - \overline{e} + 2\delta}{\epsilon}\right)\\
\leq\ & \exp\left(-\frac{\delta}{5\epsilon}\right) + \frac{4}{T_2} \exp\left(-\frac{\delta}{\epsilon}\right).
\end{aligned}$$

Finally, when u_0 is any function in D, we conclude thanks to

$$\mathbb{P}\left(u^{\epsilon,u_0}\left(\tau^{\epsilon,u_0}\right) \in N\right) \leq \mathbb{P}\left(u^{\epsilon,u_0}\left(\sigma_\rho^{\epsilon,u_0}\right) \in \partial D\right) + \sup_{u_0 \in B^0_\rho} \mathbb{P}\left(u^{\epsilon,u_0}\left(\tau^{\epsilon,u_0}\right) \in N\right)$$

and to Lemma D.3.6. □

Remark D.3.8 *Note that it has been proposed in [128] to introduce control elements in order to reduce or enhance exponentially the expected exit time or to act on the exiting points, for a limited cost. We may then think of optimizing on such external fields. However the problem is computationally involved since the optimal control problem requires double optimisation.*

D.4 Exit from a domain of attraction in H^1

We now consider a measurable bounded subset D of H^1 invariant by the flow of the deterministic equation; D and R are such that $D \subset B_R^1$. We consider both (D.2.1) and (D.2.2) where the noise is either of additive or of multiplicative type. In this section we are interested in both the fluctuation of the L^2 norm and that of the L^2 norm of the gradient. The Hamiltonian and a modified Hamiltonian will thus be of particular interest. We shall first distinguish the case where the nonlinearity is defocusing ($\lambda = -1$) where the Hamiltonian takes non negative values from the case where the nonlinearity is focusing ($\lambda = 1$) where the Hamiltonian may take negative values.

We may prove, see for example [86], that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{H}\left(\mathbf{S}(u_0,0)(t)\right) + 2\alpha\mathbf{\Psi}\left(\mathbf{S}(u_0,0)\right) = 0,$$

where $\mathbf{S}(u_0,0)$ is the solution of the deterministic weakly damped nonlinear Schrödinger equation with initial datum u_0 in H^1 and

$$\mathbf{\Psi}\left(\mathbf{S}(u_0,0)\right) = \frac{1}{2}\left\|\nabla\mathbf{S}(u_0,0)\right\|_{\mathrm{L}^2}^2 - \frac{\lambda}{2}\int_{\mathbb{R}^d}\left|\mathbf{S}(u_0,0)(x)\right|^{2\sigma+2}dx.$$

Thus, when the nonlinearity is defocusing we have

$$0 \le \mathbf{H}\left(\mathbf{S}(u_0,0)(t)\right) \le \mathbf{H}\left(u_0\right)\exp\left(-2\alpha t\right). \tag{D.4.1}$$

As it is done in [45], we shall consider in the focusing case a modified Hamiltonian denoted by $\tilde{\mathbf{H}}(u)$ defined for u in H^1 by

$$\tilde{\mathbf{H}}(u) = \mathbf{H}(u) + \beta(\sigma,d)C\left\|u\right\|_{\mathrm{L}^2}^{2+\frac{4\sigma}{2-\sigma d}}$$

where the constant C is that of the third inequality in the following sequence of inequalities where we use the Gagliardo-Nirenberg inequality

$$\frac{1}{2\sigma+2}\|u\|_{\mathrm{L}^{2\sigma+2}}^{2\sigma+2} \le C\|u\|_{\mathrm{L}^2}^{2\sigma+2-\sigma d}\|\nabla u\|_{\mathrm{L}^2}^{\sigma d} \le \frac{1}{4}\|\nabla u\|_{\mathrm{L}^2}^2 + C\|u\|_{\mathrm{L}^2}^{2+\frac{4\sigma}{2-\sigma d}},$$

and $\beta(\sigma,d) = \frac{2\sigma(2-\sigma d)}{(\sigma+2)(2-\sigma d)+2\sigma(4\sigma+3)} \vee 2$. When evaluated at the deterministic solution, the modified Hamiltonian satisfies

$$0 \le \tilde{\mathbf{H}}\left(\mathbf{S}(u_0,0)(t)\right) \le \tilde{\mathbf{H}}\left(u_0\right)\exp\left(-2\alpha\frac{3(\sigma+1)}{4\sigma+3}t\right). \tag{D.4.2}$$

Also, when the nonlinearity is defocusing we now have, for every β positive,

$$0 \leq \tilde{\mathbf{H}}\left(\mathbf{S}(u_0,0)(t)\right) \leq \tilde{\mathbf{H}}\left(u_0\right)\exp\left(-2\alpha t\right). \tag{D.4.3}$$

From the Sobolev inequalities, for ρ positive, the sets

$$\tilde{\mathbf{H}}_\rho = \left\{u \in \mathrm{H}^1 : \ \tilde{\mathbf{H}}(u) = \rho\right\} = \tilde{\mathbf{H}}^{-1}\left(\{\rho\}\right), \quad \rho > 0$$

are closed subsets of H^1 and

$$\tilde{\mathbf{H}}_{<\rho} = \left\{u \in \mathrm{H}^1 : \ \tilde{\mathbf{H}}(u) < \rho\right\} = \tilde{\mathbf{H}}^{-1}\left([0,\rho)\right) \quad \rho > 0$$

are open subsets of H^1.

Also, $\tilde{\mathbf{H}}$ is such that

$$\frac{1}{2}\|\nabla u\|_{\mathrm{L}^2}^2 + \beta C\|u\|_{\mathrm{L}^2}^{2+\frac{4\sigma}{2-\sigma d}} \leq \tilde{\mathbf{H}}(u) \leq \frac{3}{4}\|\nabla u\|_{\mathrm{L}^2}^2 + (\beta+1)C\|u\|_{\mathrm{L}^2}^{2+\frac{4\sigma}{2-\sigma d}} \tag{D.4.4}$$

when the nonlinearity is defocusing and

$$\frac{1}{4}\|\nabla u\|_{\mathrm{L}^2}^2 + C\|u\|_{\mathrm{L}^2}^{2+\frac{4\sigma}{2-\sigma d}} \leq \tilde{\mathbf{H}}(u) \leq \frac{1}{2}\|\nabla u\|_{\mathrm{L}^2}^2 + \beta(\sigma,d)C\|u\|_{\mathrm{L}^2}^{2+\frac{4\sigma}{2-\sigma d}} \tag{D.4.5}$$

when it is focusing. Thus the sets $\tilde{\mathbf{H}}_{<\rho}$ for ρ positive are bounded in H^1 and a bounded set in H^1 is bounded for $\tilde{\mathbf{H}}$.

We will no longer distinguish the focusing and defocusing cases and will take the same value of β, *i.e.* $\beta(\sigma,d)$. Also to simplify the notations we will sometimes drop the dependence of the solution in ϵ and u_0.

The fluctuation of $\tilde{\mathbf{H}}\left(u^{\epsilon,u_0}(t)\right)$ is of particular interest. We have the following result when the noise is of additive type.

Proposition D.4.1 *When u denotes the solution of equation* (D.2.1), $(e_j)_{j\in\mathbb{N}}$ *a complete orthonormal system of* L^2*, the following decomposition holds*

$$\begin{aligned}
\tilde{\mathbf{H}}\left(u(t)\right) &= \tilde{\mathbf{H}}\left(u_0\right) \\
&\quad -2\alpha\int_0^t \mathbf{\Psi}\left(u(s)\right)ds - 2\beta C\left(1+\tfrac{2\sigma}{2-\sigma d}\right)\alpha\int_0^t \|u(s)\|_{\mathrm{L}^2}^{2+\frac{4\sigma}{2-\sigma d}}ds \\
&\quad +\sqrt{\epsilon}\Big(\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t \nabla\overline{u}(s)\nabla dW(s)dx - \lambda\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t |u(s)|^{2\sigma}\,\overline{u}(s)dW(s)dx \\
&\qquad\qquad +2\beta C\left(1+\tfrac{2\sigma}{2-\sigma d}\right)\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t \|u(s)\|_{\mathrm{L}^2}^{\frac{4\sigma}{2-\sigma d}}\,\overline{u}(s)dW(s)dx\Big) \\
&\quad -\tfrac{\lambda\epsilon}{2}\sum_{j\in\mathbb{N}}\int_0^t\int_{\mathbb{R}^d}\left[|u(s)|^{2\sigma}|\Phi e_j|^2 + 2\sigma|u(s)|^{2\sigma-2}\left(\mathfrak{Re}(\overline{u}(s)\Phi e_j)\right)^2\right]dxds \\
&\quad +\tfrac{\epsilon}{2}\|\nabla\Phi\|_{\mathcal{L}_2^{0,0}}^2 t + \epsilon\beta C\left(1+\tfrac{2\sigma}{2-\sigma d}\right)\|\Phi\|_{\mathcal{L}_2^{0,0}}^2\int_0^t \|u(s)\|_{\mathrm{L}^2}^{\frac{4\sigma}{2-\sigma d}}ds \\
&\quad +\epsilon\beta C\tfrac{4\sigma}{2-\sigma d}\left(1+\tfrac{2\sigma}{2-\sigma d}\right)\sum_{j\in\mathbb{N}}\int_0^t \|u(s)\|_{\mathrm{L}^2}^{2\left(\frac{2\sigma}{2-\sigma d}-1\right)}\left(\mathfrak{Re}\int_{\mathbb{R}^d}\overline{u}(s)\Phi e_j dx\right)^2 ds
\end{aligned}$$

Proof. The result follows from the Itô formula. The main difficulty is in justifying the computations. We may proceed as in [37]. □

Also, when the noise is of multiplicative type we obtain the following proposition.

Proposition D.4.2 *When u denotes the solution of equation* (D.2.1), $(e_j)_{j\in\mathbb{N}}$ *a complete orthonormal system of* L^2, *the following decomposition holds*

$$\begin{aligned}\tilde{\mathbf{H}}(u(t)) = \; & \tilde{\mathbf{H}}(u_0)\\ & -2\alpha\int_0^t \mathbf{\Psi}(u(s))\,ds - 2\beta C\left(1+\frac{2\sigma}{2-\sigma d}\right)\alpha\int_0^t \|u(s)\|_{\mathrm{L}^2}^{2+\frac{4\sigma}{2-\sigma d}}ds\\ & +\sqrt{\epsilon}\,\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t u(s)\nabla\overline{u}(s)\nabla dW(s)dx\\ & +\frac{\epsilon}{2}\sum_{j\in\mathbb{N}}\int_0^t\int_{\mathbb{R}^d}|u(s)|^2|\nabla\Phi e_j|^2dxds.\end{aligned}$$

The first exit time τ^{ϵ,u_0} from the domain D in H^1 is defined as in Section D.2. Note that the domain D may be a domain of attraction of the form $\tilde{\mathbf{H}}_{<a}$ where a is positive. We also define

$$\overline{e} = \inf\left\{I_T^0(w):\ w(T)\in\overline{D}^c,\ T>0\right\},$$

and for ρ positive small enough

$$e_\rho = \inf\left\{I_T^{u_0}(w):\ \tilde{\mathbf{H}}(u_0)\le\rho,\ w(T)\in(D_{-\rho})^c,\ T>0\right\},$$

where $D_{-\rho}=D\setminus\mathcal{N}^1(\partial D,\rho)$. Then we set

$$\underline{e} = \lim_{\rho\to 0} e_\rho.$$

Also, for ρ positive small enough, N a closed subset of the boundary of D, we define

$$e_{N,\rho} = \inf\left\{I_T^{u_0}(w):\ \tilde{\mathbf{H}}(u_0)\le\rho,\ w(T)\in\left(D\setminus\mathcal{N}^1(N,\rho)\right)^c,\ T>0\right\}$$

and

$$\underline{e}_N = \lim_{\rho\to 0} e_{N,\rho}.$$

We finally also introduce

$$\sigma_\rho^{\epsilon,u_0} = \inf\left\{t\ge 0: u^{\epsilon,u_0}(t)\in\tilde{\mathbf{H}}_{<\rho}\cup D^c\right\},$$

where $\tilde{\mathbf{H}}_{<\rho}\subset D$.

Again we have the following inequalities.

Lemma D.4.3 $0 < \underline{e} \leq \overline{e}$.

Proof. We only have to prove the first inequality. Integrating the equation describing the evolution of $\tilde{\mathbf{H}}\left(\mathbf{S}\left(u_0,h\right)(t)\right)$ via the Duhamel formula where the skeleton is that of the equation with an additive noise we obtain

$$\begin{array}{l}\tilde{\mathbf{H}}\left(\mathbf{S}(u_0,h)(T)\right) - \exp\left(-2\alpha\frac{3(\sigma+1)}{4\sigma+3}T\right)\tilde{\mathbf{H}}\left(u_0\right)\\ \leq \int_0^T \exp\left(-2\alpha\frac{3(\sigma+1)}{4\sigma+3}(T-s)\right)\left[\mathfrak{Im}\int_{\mathbb{R}^d}\left(\nabla\mathbf{S}(u_0,h)\nabla\overline{\Phi h}\right)(s,x)dx\right.\\ \quad -\lambda\mathfrak{Im}\int_{\mathbb{R}^d}\left(|\mathbf{S}(u_0,h)|^{2\sigma}\mathbf{S}(u_0,h)\overline{\Phi h}\right)(s,x)dx\\ \quad \left.-2C\beta\left(1+\frac{2\sigma}{2-\sigma d}\right)\mathfrak{Im}\int_{\mathbb{R}^d}\left(\mathbf{S}\left(u_0,h\right)\overline{\Phi h}\right)(s,x)dx\right]ds,\end{array}$$

with a focusing or defocusing nonlinearity. Let d denote the positive distance between 0 and ∂D. Take ρ such that the distance between B^1_ρ and $\left(D_{-\rho}\right)^c$ is larger than $\frac{d}{2}$. We then have, from the fact that the Sobolev injection from H^1 into $\mathrm{L}^{2\sigma+2}$,

$$\begin{array}{l}\frac{d}{2} \leq \int_0^T \exp\left(-2\alpha\frac{3(\sigma+1)}{4\sigma+3}(T-s)\right)\left[R\|\Phi\|_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{H}^1)}\|h\|_{\mathrm{L}^2}\right.\\ \quad +CR^{2\sigma+1}\|\Phi\|_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{H}^1)}\|h\|_{\mathrm{L}^2}\\ \quad \left.+2C\beta\left(1+\frac{2\sigma}{2-\sigma d}\right)R\|\Phi\|_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{L}^2)}\|h\|_{\mathrm{L}^2}\right]ds,\end{array}$$

We conclude as in Lemma D.3.1 and use that from the choice of β the complementary of a ball is included in the complementary of a set $\tilde{\mathbf{H}}_{<a}$. In the case of the skeleton of the equation with a multiplicative noise, it is enough to replace the term in bracket in the right hand side of the above formula by $\mathfrak{Im}\int_{\mathbb{R}^d}\left(\nabla\mathbf{S}\left(u_0,h\right)\overline{\mathbf{S}\left(u_0,h\right)\nabla\Phi h}\right)(s,x)dx$. Recall that we can proceed as in the additive case since we have imposed that Φ belongs to $\mathcal{L}^{0,s}_{2,\mathbb{R}}$ where $s > \frac{d}{2}+1$, in particular Φ belongs to $\mathcal{L}_c\left(\mathrm{L}^2,\mathrm{W}^{1,\infty}\right)$. □

The theorems of Section D.2 still hold for a domain of attraction in H^1 and a noise of additive and multiplicative type.

Theorem D.4.4 *For every u_0 in D and δ positive, there exists L positive such that*

$$\overline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(\tau^{\epsilon,u_0}\notin\left(\exp\left(\frac{\underline{e}-\delta}{\epsilon}\right),\exp\left(\frac{\overline{e}+\delta}{\epsilon}\right)\right)\right)\leq -L, \qquad (D.4.6)$$

and for every u_0 in D,

$$\underline{e}\leq\underline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{E}\left(\tau^{\epsilon,u_0}\right)\leq\overline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{E}\left(\tau^{\epsilon,u_0}\right)\leq\overline{e}. \qquad (D.4.7)$$

Moreover, for every δ positive, there exists L positive such that

$$\overline{\lim}_{\epsilon\to 0}\epsilon\log\sup_{u_0\in D}\mathbb{P}\left(\tau^{\epsilon,u_0}\geq\exp\left(\frac{\overline{e}+\delta}{\epsilon}\right)\right)\leq -L, \tag{D.4.8}$$

and

$$\overline{\lim}_{\epsilon\to 0}\epsilon\log\sup_{u_0\in D}\mathbb{E}\left(\tau^{\epsilon,u_0}\right)\leq\overline{e}. \tag{D.4.9}$$

Theorem D.4.5 *If $\underline{e}_N>\overline{e}$, then for every u_0 in D, there exists L positive such that*

$$\overline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(u^{\epsilon,u_0}\left(\tau^{\epsilon,u_0}\right)\in N\right)\leq -L.$$

Again we may deduce the corollary

Corollary D.4.6 *Assume that v^* in ∂D is such that for every δ positive and $N=\{v\in\partial D:\ \|v-v^*\|_{\mathrm{L}^2}\geq\delta\}$ we have $\underline{e}_N>\overline{e}$ then*

$$\forall\delta>0,\ \forall u_0\in D,\ \exists L>0:\ \overline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(\left\|u^{\epsilon,u_0}\left(\tau^{\epsilon,u_0}\right)-v^*\right\|_{\mathrm{L}^2}\geq\delta\right)\leq -L.$$

The proof of these results still relies on three lemmas and the uniform LDP. Let us now state the lemmas for both a noise of additive and of multiplicative type.

Lemma D.4.7 *For every ρ and L positive with $\tilde{\mathbf{H}}_{<\rho}\subset D$, there exists T and ϵ_0 positive such that for every u_0 in D and ϵ in $(0,\epsilon_0)$,*

$$\mathbb{P}\left(\sigma_\rho^{\epsilon,u_0}>T\right)\leq\exp\left(-\frac{L}{\epsilon}\right).$$

Proof. We proceed as in the proof of Lemma D.3.5.
Let d denote the positive distance between 0 and $D\setminus\tilde{\mathbf{H}}_{<\rho}$. Take α positive such that $\alpha\rho<d$. The domain D is uniformly attracted to 0, thus there exists a time T_1 such that for every initial datum u_1 in $\mathcal{N}^1\left(D\setminus\tilde{\mathbf{H}}_{<\rho},\frac{\alpha\rho}{8}\right)$, for $t\geq T_1$, $\mathbf{S}\left(u_1,0\right)(t)$ belongs to $B^1_{\frac{\alpha\rho}{8}}$.

We could also prove, see [37], that there exists a constant M' which depends on T_1, R, σ and α such that

$$\sup_{u_1\in\mathcal{N}^1\left(D\setminus\tilde{\mathbf{H}}_{<\rho},\frac{\alpha\rho}{8}\right)}\left\|\mathbf{S}\left(u_1,0\right)\right\|_{X^{(T_1,2\sigma+2)}}\leq M'. \tag{D.4.10}$$

The Step 2, corresponding to that of Lemma D.3.5, in the proof in the additive case uses the truncation argument, upper bounds similar to that in [37] derived from the Strichartz inequalities on smaller intervals; we shall also replace in the proof of Lemma D.3.5 $\frac{\rho}{8}$ by $\frac{\alpha\rho}{8}$.

In Step 2 for the multiplicative case, we also introduce the truncation in front of the term $u\Phi h$ in the controlled PDE.

The end of the proof is identical to that of Lemma D.3.5, the LDP is the LDP in $\mathrm{C}\left([0,T];\mathrm{H}^1\right)$, for additive or multiplicative noises. □

Lemma D.4.8 *For every ρ positive such that $\tilde{\mathbf{H}}_\rho \subset D$ and u_0 in D, there exists L positive such that*

$$\overline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(u^{\epsilon,u_0}\left(\sigma_\rho^{\epsilon,u_0}\right) \in \partial D\right) \le -L$$

Proof. It is the same proof as for Lemma D.3.6. We only have to replace $B^0_{\frac{\rho}{2}}$ by any ball in H^1 centered at 0 and included in $\tilde{\mathbf{H}}_{<\rho}$ and use the LDP in $\mathrm{C}\left([0,T];\mathrm{H}^1\right)$. □

Lemma D.4.9 *For every ρ and L positive such that $\tilde{\mathbf{H}}_{2\rho} \subset D$, there exists $T(L,\rho) < \infty$ such that*

$$\overline{\lim}_{\epsilon\to 0}\epsilon \log \sup_{u_0\in\tilde{\mathbf{H}}_\rho} \mathbb{P}\left(\sup_{t\in[0,T(L,\rho)]}\left(\tilde{\mathbf{H}}\left(u^{\epsilon,u_0}(t)\right) - \tilde{\mathbf{H}}\left(u_0\right)\right) \ge \rho\right) \le -L$$

Proof. Integrating the Itô differential relation using the Duhamel formula allows to get rid of the drift term that is not originated from the bracket. Indeed, the event

$$\left\{\sup_{t\in[0,T(L,\rho)]}\left(\tilde{\mathbf{H}}\left(u^{\epsilon,u_0}(t)\right) - \tilde{\mathbf{H}}\left(u_0\right)\right) \ge \rho\right\}$$

is included in

$$\left\{\sup_{t\in[0,T(L,\rho)]}\left(\tilde{\mathbf{H}}\left(u^{\epsilon,u_0}(t)\right) - \exp\left(-2\alpha\left(\frac{3(\sigma+1)}{4\sigma+3}\right)T(L,\rho)\right)\tilde{\mathbf{H}}\left(u_0\right)\right) \ge \rho\right\}.$$

Then, setting $c(\sigma)=\frac{3(\sigma+1)}{4\sigma+3}$ and $m(\sigma,d)=1+\frac{2\sigma}{2-\sigma d}$, dropping the exponents ϵ and u_0 to have more concise formulas, we obtain in the additive case

$$
\begin{aligned}
&\tilde{\mathbf{H}}\left(u(t)\right)-\exp\left(-2\alpha c(\sigma)t\right)\tilde{\mathbf{H}}\left(u_0\right)\\
&\leq\sqrt{\epsilon}\Big(\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t\exp\left(-2\alpha c(\sigma)(t-s)\right)\nabla\overline{u}(s)\nabla dW(s)dx\\
&-\lambda\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t\exp\left(-2\alpha c(\sigma)(t-s)\right)|u(s)|^{2\sigma}\overline{u}(s)dW(s)dx\\
&+2\beta Cm(\sigma,d)\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t\exp\left(-2\alpha c(\sigma)(t-s)\right)\|u(s)\|_{\mathrm{L}^2}^{\frac{4\sigma}{2-\sigma d}}\overline{u}(s)dW(s)dx\Big)\\
&-\frac{\lambda\epsilon}{2}\sum_{j\in\mathbb{N}}\int_0^t\exp\left(-2\alpha c(\sigma)(t-s)\right)\int_{\mathbb{R}^d}\Big[|u(s)|^{2\sigma}|\Phi e_j|^2\\
&\qquad\qquad+2\sigma|u(s)|^{2\sigma-2}\left(\mathfrak{Re}(\overline{u}(s)\Phi e_j)\right)^2\Big]dxds\\
&+\frac{\epsilon}{4\alpha c(\sigma)}\left(1-\exp\left(-2\alpha c(\sigma)t\right)\right)\|\nabla\Phi\|_{\mathcal{L}_2^{0,0}}^2\\
&+\epsilon\beta Cm(\sigma,d)\|\Phi\|_{\mathcal{L}_2^{0,0}}^2\int_0^t\exp\left(-2\alpha c(\sigma)(t-s)\right)\|u(s)\|_{\mathrm{L}^2}^{\frac{4\sigma}{2-\sigma d}}ds\\
&+\epsilon\beta C\frac{4\sigma}{2-\sigma d}m(\sigma,d)\sum_{j\in\mathbb{N}}\int_0^t\exp\left(-2\alpha c(\sigma)(t-s)\right)\|u(s)\|_{\mathrm{L}^2}^{2\left(\frac{2\sigma}{2-\sigma d}-1\right)}\left(\mathfrak{Re}\int_{\mathbb{R}^d}\overline{u}(s)\Phi e_jdx\right)^2ds.
\end{aligned}
$$

We again use a localization argument and replace the process u by the process u^τ stopped at the first exit time off $\tilde{\mathbf{H}}_{<2\rho}$. We use (D.4.4) and (D.4.5) and obtain

$$
\|u^\tau\|_{\mathrm{H}^1}^2\leq 8\rho+\left(\frac{2\rho}{C\sigma}\right)^{\frac{1}{1+\frac{2\sigma}{2-\sigma d}}}.
$$

We denote the right hand side of the above by $b(\rho,\sigma,d)$.
From the Hölder inequality along with the Sobolev injection of H^1 into $\mathrm{L}^{2\sigma+2}$ we obtain the following upper bound for the drift

$$
\begin{aligned}
&\frac{\epsilon}{4\alpha c(\sigma)}\left[(1+2\sigma)c(1,2\sigma+2)^{2\sigma+2}\|\Phi\|_{\mathcal{L}_2^{0,1}}^2b(\rho,\sigma,d)^{2\sigma}+\|\nabla\Phi\|_{\mathcal{L}_2^{0,0}}^2\right]\\
&+\frac{\epsilon\beta C}{2\alpha c(\sigma)}m(\sigma,d)\left(1+\frac{4\sigma}{2-\sigma d}\right)\|\Phi\|_{\mathcal{L}_2^{0,0}}^2b(\rho,\sigma,d)^{\frac{4\sigma}{2-\sigma d}}
\end{aligned}
$$

where we denote by $c(1,2\sigma+2)$ the norm of the continuous injection of H^1 into $\mathrm{L}^{2\sigma+2}$.
Thus, choosing ϵ small enough, it is enough to show the result for the stochastic integral remplacing ρ by $\frac{\rho}{2}$. Also it is enough to show the result for each of the three stochastic integrals replacing $\frac{\rho}{2}$ by $\frac{\rho}{6}$. With the same one parameter families and similar computations as in the proof of Lemma D.3.7, we know that it is enough to obtain upper bounds of the brackets of the stochastic integrals

$$
\begin{aligned}
&Z_1(t)=\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t\exp\left(2\alpha c(\sigma)s\right)\nabla\overline{u^\tau}(s)\nabla dW(s)dx\\
&Z_2(t)=\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t\exp\left(2\alpha c(\sigma)s\right)|u^\tau(s)|^{2\sigma}\overline{u^\tau}(s)dW(s)dx\\
&Z_3(t)=2\beta Cm(\sigma,d)\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t\exp\left(2\alpha c(\sigma)s\right)\|u^\tau(s)\|_{\mathrm{L}^2}^{\frac{4\sigma}{2-\sigma d}}\overline{u^\tau}(s)dW(s)dx.
\end{aligned}
$$

We then obtain

$$\begin{aligned}
&d < Z_1 >_t \leq \exp\left(4\alpha c(\sigma)t\right)\sum_{j\in\mathbb{N}}\left(\nabla u^\tau(t), -i\nabla\Phi e_j\right)_{\mathrm{L}^2}^2 dt\\
&d < Z_2 >_t \leq \exp\left(4\alpha c(\sigma)t\right)\sum_{j\in\mathbb{N}}\left(|u^\tau(t)|^{2\sigma}u^\tau(t), -i\Phi e_j\right)_{\mathrm{L}^2}^2 dt\\
&d < Z_3 >_t \leq 4\beta^2C^2m(\sigma,d)^2\exp\left(4\alpha c(\sigma)t\right)\|u^\tau(t)\|_{\mathrm{L}^2}^{\frac{8\sigma}{2-\sigma d}}\sum_{j\in\mathbb{N}}\left(u^\tau(t), -i\Phi e_j\right)_{\mathrm{L}^2}^2 dt.
\end{aligned}$$

Using the Hölder inequality and, for Z_2, the continuous Sobolev injection of H^1 into $\mathrm{L}^{2\sigma+2}$ we obtain

$$\begin{aligned}
&d < Z_1 >_t \leq \exp\left(4\alpha c(\sigma)t\right)\|\Phi\|_{\mathcal{L}_2^{0,1}}^2 b(\rho,\sigma,d)dt\\
&d < Z_2 >_t \leq \exp\left(4\alpha c(\sigma)t\right)c(1,2\sigma+2)^{2(2\sigma+2)}\|\Phi\|_{\mathcal{L}_2^{0,1}}^2 b(\rho,\sigma,d)^{2\sigma+1}dt\\
&d < Z_3 >_t \leq 4\beta^2C^2m(\sigma,d)^2\exp\left(4\alpha c(\sigma)t\right)b(\rho,\sigma,d)^{\left(1+\frac{4\sigma}{2-\sigma d}\right)}\|\Phi\|_{\mathcal{L}_2^{0,1}}^2 dt.
\end{aligned}$$

We can then bound each of the three remainders $\left(R_l^i(t)\right)_{i=1,2,3}$ similar to that of Lemma D.3.7 using the inequality $R_l^i(t) \leq 3l\int_0^t d < Z_i >_t$.
We conclude that it is possible to choose $T(L,\rho)$ equal to

$$\frac{1}{4\alpha c(\sigma)}\log\left(\frac{\alpha c(\sigma)\rho^2}{90b(\rho,\sigma,d)\|\Phi\|_{\mathcal{L}_2^{0,1}}^2\max\left(1, c(1,2\sigma+2)^{2(2\sigma+1)}b(\rho,\sigma,d)^{2\sigma}, 4\beta^2C^2m(\sigma,d)^2b(\rho,\sigma,d)^{\frac{4\sigma}{2-\sigma d}}\right)}\right).$$

When the noise is of multiplicative type we obtain

$$\begin{aligned}
&\tilde{\mathbf{H}}\left(u(t)\right) - \exp\left(-2\alpha c(\sigma)t\right)\tilde{\mathbf{H}}\left(u_0\right)\\
&\leq \sqrt{\epsilon}\,\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t \exp\left(-2\alpha c(\sigma)(t-s)\right)u(s)\nabla\overline{u}(s)\nabla dW(s)dx\\
&\quad+\frac{\epsilon}{2}\sum_{j\in\mathbb{N}}\int_0^t \exp\left(-2\alpha c(\sigma)(t-s)\right)\int_{\mathbb{R}^d}|u(s)|^2|\nabla\Phi e_j|^2 dxds.
\end{aligned}$$

Again we use a localization argument and consider the process u stopped at the exit off $\tilde{\mathbf{H}}_{2\rho}$. As Φ is Hilbert-Schmidt from L^2 into $\mathrm{H}^s_{\mathbb{R}}$, the second term of the right hand side is less than $\frac{\epsilon}{4\alpha c(\sigma)}\|\Phi\|_{\mathcal{L}_2^{0,s}}^2 b(\rho,\sigma,d)$ and for ϵ small enough, it is enough to prove the result for the stochastic integral replacing ρ by $\frac{\rho}{2}$. We know that it is enough to obtain an upper bound of the bracket of

$$Z(t) = \mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t \exp\left(2\alpha c(\sigma)s\right)u^\tau(s)\nabla\overline{u}^\tau(s)\nabla dW(s)dx.$$

We obtain

$$d < Z >_t \leq \exp\left(4\alpha c(\sigma)t\right)\sum_{j\in\mathbb{N}}\left(\nabla u^\tau(t), -iu^\tau(t)\nabla\Phi e_j\right)_{\mathrm{L}^2}^2 dt.$$

Denoting by $c(s,\infty)$ the norm of the Sobolev injection of $\mathrm{H}^s_{\mathbb{R}}$ into $\mathrm{W}^{1,\infty}_{\mathbb{R}}$ we deduce that

$$d < Z >_t \leq \exp\left(4\alpha c(\sigma)t\right)c(s,\infty)^2\|\Phi\|_{\mathcal{L}_2^{0,s}}^2 b(\rho,\sigma,d)^2 dt.$$

Finally, we conclude that we may choose

$$T(L,\rho)=\frac{1}{4\alpha c(\sigma)}\log\left(\frac{\alpha c(\sigma)\rho^2}{10b(\rho,\sigma,d)^2c(s,\infty)^2\|\Phi\|^2_{\mathcal{L}_2^{0,s}}L}\right).$$

□

We may now prove Theorem D.4.6 and D.4.7.
Here are some of the specific aspects of the proof of Theorem D.4.6.
Proof of Theorem D.4.6. There is no difference in the proof of the upper bound on τ^{ϵ,u_0}. Let us thus focus on the lower bound. Take δ positive. Since $\underline{e}>0$, we now choose ρ positive such that $\underline{e}-\frac{\delta}{4}\le e_\rho$, $\tilde{\mathbf{H}}_{2\rho}\subset D$ and $\tilde{\mathbf{H}}_{2\rho}\subset D^c_{-\rho}$. We define the sequences of stopping times $\theta_0=0$ and for k in $\mathbb{N}$,

$$\begin{aligned}\tau_k&=\inf\left\{t\ge\theta_k:\ u^{\epsilon,u_0}(t)\in\tilde{\mathbf{H}}_{<\rho}\cup D^c\right\},\\ \theta_{k+1}&=\inf\left\{t>\tau_k:\ u^{\epsilon,u_0}(t)\in\tilde{\mathbf{H}}_{2\rho}\right\},\end{aligned}$$

where $\theta_{k+1}=\infty$ if $u^{\epsilon,u_0}(\tau_k)\in\partial D$. Let us fix $T_1=T\left(\underline{e}-\frac{3\delta}{4},\rho\right)$ given by Lemma D.4.9. We now use that for u_0 in D and m a positive integer,

$$\begin{aligned}\mathbb{P}\left(\tau^{\epsilon,u_0}\le mT_1\right)\le\ &\mathbb{P}\left(\tau^{\epsilon,u_0}=\tau_0\right)+\textstyle\sum_{k=1}^m\mathbb{P}\left(\tau^{\epsilon,u_0}=\tau_k\right)\\ &+\textstyle\sum_{k=1}^m\mathbb{P}\left(\theta_k-\tau_{k-1}\le T_1\right)\end{aligned}\qquad\text{(D.4.11)}$$

and conclude as in the proof of Theorem D.3.2. □

We may also check that the proof of Theorem D.3.3 also applies to prove Theorem D.4.5, the LDPs are those in H^1 and the sequences of stopping times are those defined above.

Again the control argument to prove that $\underline{e}=\overline{e}$ seems difficult. We may however apply in the H^1 case the Sobolev injection in order to treat the nonlinearity.

Let us now make an interesting comment. Assume that we are able to prove Theorem D.4.4 with $\overline{e}=\underline{e}$ at least for an additive noise. The exit points are then characterized by the infimum of the quasi potential on the boundary of the domain of attraction. Under assumptions such that Φ commutes with the Laplacian and that Φ does not change the phase, we have an explicit expression of the quasi potential since the vector-field in the drift is the sum of a gradient vector-field and a vector-field which is orthogonal to the first one, see for example [68, 73]. These assumptions on Φ are such that we can mimick the computations for the ideal white noise.

The quasipotential is proportional to $\mathbf{N}_H(u)=\left\|\left(\Phi_{|\mathrm{Ker}\Phi^\perp}\right)^{-1}u\right\|_{\mathrm{L}^2}^2$. Indeed, the rate function of the LDP applied to u, for T positive, may be written for γ in $\mathbb{R}$,

$$\begin{aligned}
&I_T^{u_0}(u)\\
&=\tfrac{1}{2}\int_0^T\left\|\left(\left(\Phi_{|\mathrm{Ker}\Phi^\perp}\right)^{-1}\left(i\tfrac{\partial u}{\partial t}+i\alpha(1-\gamma)u+i\alpha\gamma u-\Delta u-\lambda|u|^{2\sigma}u\right)\right)(s)\right\|_{\mathrm{L}^2}^2 ds\\
&=\tfrac{1}{2}\int_0^T\left\|\left(\left(\Phi_{|\mathrm{Ker}\Phi^\perp}\right)^{-1}\left(i\tfrac{\partial u}{\partial t}+i\alpha(1-\gamma)u-\Delta u-\lambda|u|^{2\sigma}u\right)\right)(s)\right\|_{\mathrm{L}^2}^2 ds\\
&\quad+\tfrac{\alpha\gamma}{2}\left[\mathbf{N}_H(u(T))-\mathbf{N}_H(u_0)\right]+\alpha^2\left(\tfrac{\gamma^2}{2}+(1-\gamma)\gamma\right)\int_0^T\mathbf{N}_H(u(s))ds.
\end{aligned}$$

The last term is equal to zero if and only if $\gamma=2$ or $\gamma=0$. When $\gamma=2$ we obtain

$$\begin{aligned}
V(0,u_f)=\ &\inf\Big\{\tfrac{1}{2}\int_0^T\left\|\left(\left(\Phi_{|\mathrm{Ker}\Phi^\perp}\right)^{-1}\left(\tfrac{\partial u}{\partial t}-\alpha u+i\Delta u+i\lambda|u|^{2\sigma}u\right)\right)(s)\right\|_{\mathrm{L}^2}^2 ds\\
&\qquad+\alpha\mathbf{N}_H(u_f):\ u(0)=0,\ u(T)=u_f,\ T>0\Big\}\\
\geq\ &\alpha\mathbf{N}_H(u_f).
\end{aligned}$$

In order to prove the converse inequality, we should prove that there exists a sequence of functions satisfying the boundary conditions such that the first term is arbitrarily small; it is another control problem. Assume that we are able to solve it, then the quasi potential is indeed proportional to the mass.

Suppose now that the domain of attraction is a set of the form $\tilde{\mathbf{H}}_{<\rho}$ for ρ positive. Exit points are points of the level set $\tilde{\mathbf{H}}_\rho$ that minimize $\mathbf{N}_H$. Since $\mathbf{N}_H$ is also the square of the norm of the reproducing kernel Hilbert space of the law of $W(1)$, or because Φ is Hilbert-Schmidt, we know that infima do exist. Also because they satisfy $\tilde{\mathbf{H}}_{<\rho}(u)=\rho$ they are different from 0. Note that in the ideal white noise case infima do not exist and the infimum is 0. By a standard minimization argument we deduce that the exit points satisfy for some ω in $\mathbb{R}$,

$$\left(\left(\left(\Phi_{|\mathrm{Ker}\Phi^\perp}\right)^{-1}\right)^*\left(\Phi_{|\mathrm{Ker}\Phi^\perp}\right)^{-1}+\omega 2\beta C\|u\|_{\mathrm{L}^2}^{\frac{4\sigma}{2-\sigma d}}\right)u=\omega\left(\Delta u+\lambda|u|^{2\sigma}u\right).$$

The case where $\omega=0$ corresponds to $u=0$; we may thus assume that $\omega\neq 0$. When $\Phi=I$ and $\lambda=1$, this equation has solutions which are solitary waves profiles.

If we could approximate the white noise in a suitable sense and justify all of the above rigorously, it would give an important information on the dynamical behavior of the solutions of the nonlinear equation under the influence of a noise. Indeed, it would give an indication that the energy injected by the noise organizes and creates solitary waves. Note that such behavior has been observed numerically in [46] on the Korteweg-de Vries equation.

D.5 Annex - proof of Theorem D.2.1

The following lemma proves to be at the core of the proof of the uniform LDPs. It is often called Azencott lemma or Freidlin-Wentzell inequality. The differences with the result of [82] are that here the initial data are the same for the random process and the skeleton and that the "for every ρ positive" stands before "there exists ϵ_0 and γ positive". We shall only stress on the differences in the proof.

Lemma D.5.1 *For every a, L, T, δ and ρ positive, f in C_a, p in $\mathcal{A}(d)$, there exists ϵ_0 and γ positive such that for every ϵ in $(0,\epsilon_0)$, $\|u_0\|_{\mathrm{H}^1} \leq \rho$,*

$$\epsilon \log \mathbb{P}\left(\left\|u^{\epsilon,u_0} - \tilde{\mathbf{S}}(u_0,f)\right\|_{X^{(T,p)}} \geq \delta;\ \|\sqrt{\epsilon}W - f\|_{\mathrm{C}\left([0,T];\mathrm{H}^s_{\mathbb{R}}\right)} < \gamma\right) \leq -L.$$

Proof. There are still three steps in the proof of this result. The first step is a change of measure to center the process around f. It uses the Girsanov theorem and is the same as in [82].
The second step is a reduction to estimates for the stochastic convolution. It strongly involves the Strichartz inequalities but it is slightly different than in [82]. The truncation argument has to hold for all $\|u_0\|_{\mathrm{H}^1} \leq \rho$. Thus we use the fact that there exists $M = M(T,\rho,\sigma)$ positive such that

$$\sup_{u_1 \in B^1_\rho} \left\|\tilde{\mathbf{S}}(u_1,f)\right\|_{X^{(T,p)}} \leq M.$$

The proof of this fact follows from the computations in [37], we will recall the arguments in L^2 in the proof of Lemma D.3.5. The result in H^1 will again be used in the proof of Lemma D.4.7. As the initial data are the same for the random process and the skeleton, the remaining of the argument does not require restrictions on ρ.
The third step corresponds to estimates for the stochastic convolution. It is the same as in [82].
Note that the extra damping term in the drift is treated easily thanks to the Strichartz inequalities. □

We shall now prove Theorem D.2.1.

Proof of Theorem D.2.1. Let us start with the case of an additive noise. Recall that, in that case, the mild solution of the stochastic equation could be written as a function of the perturbation in the convolution form. Let $v^{u_0}(Z)$ denote the solution of

$$\begin{cases} i\frac{\partial v}{\partial t} - \left(\Delta v + |v-iZ|^{2\sigma}(v-iZ) - i\alpha(v-iZ)\right) = 0, \\ v(0) = u_0, \end{cases}$$

or equivalently a fixed point of the functional $\mathcal{F}_Z$ such that

$$\begin{aligned} \mathcal{F}_Z(v)(t) = \; & U(t)u_0 - i\lambda \int_0^t U(t-s)\left(|(v-iZ)(s)|^{2\sigma}(v-iZ)(s)\right)ds \\ & -\alpha\int_0^t U(t-s)(v-iZ)(s)ds, \end{aligned}$$

where Z belongs to $\mathrm{C}\left([0,T];\mathrm{L}^2\right)$ (respectively $\mathrm{C}\left([0,T];\mathrm{H}^1\right)$). If u^{ϵ,u_0} is defined as $u^{\epsilon,u_0} = v^{u_0}(Z^\epsilon) - iZ^\epsilon$ where Z^ϵ is the stochastic convolution $Z^\epsilon(t) = \sqrt{\epsilon}\int_0^t U(t-s)dW(s)$ then u^{ϵ,u_0} is a solution of the stochastic equation. Consequently, if $\mathcal{G}(\cdot,u_0)$ denotes the mapping from $\mathrm{C}\left([0,T];\mathrm{L}^2\right)$ (respectively $\mathrm{C}\left([0,T];\mathrm{H}^1\right)$) to $\mathrm{C}\left([0,T];\mathrm{L}^2\right)$ (respectively $\mathrm{C}\left([0,T];\mathrm{H}^1\right)$) defined by $\mathcal{G}(Z,u_0) = v^{u_0}(Z) - iZ$, we obtain $u^{\epsilon,u_0} = \mathcal{G}(Z^\epsilon,u_0)$. We may also check with arguments similar to that of [37, 81], involving the Strichartz inequalities that the mapping $\mathcal{G}$ is equicontinuous in its first arguments for second arguments in bounded sets of L^2 (respectively H^1). The result now follows from Proposition 5 in [131].

Let us now consider the case of a multiplicative noise. Initial data belong to H^1 and we consider paths in H^1. The proof is very close to that in [82].

The main tool is again the Azencott lemma or almost continuity of the Itô map. We need the slightly different result from that in [82].

Let us see how the above lemma implies (i) and (ii).

We start with the upper bound (i). Take a, ρ, T and δ positive. Take $L > a$. For $\tilde{a}$ in $(0,a]$, we denote by

$$A_{\tilde{a}}^{u_0} = \left\{ v \in \mathrm{C}\left([0,T];\mathrm{H}^1\right) : \; d_{\mathrm{C}([0,T];\mathrm{H}^1)}\left(v, K_T^{u_0}(\tilde{a})\right) \geq \delta \right\}.$$

Note that we have $A_a^{u_0} \subset A_{\tilde{a}}^{u_0}$ and $C_{\tilde{a}} \subset C_a$. Take $\tilde{a} \in (0,a]$ and f such that $I_T^W(f) < \tilde{a}$.

We shall now apply the Azencott lemma and choose $p=2$. We obtain $\epsilon_{\rho,f,\delta}$ and $\gamma_{\rho,f,\delta}$ positive such that for every $\epsilon \leq \epsilon_{\rho,f,\delta}$ and u_0 such that $\|u_0\|_{\mathrm{H}^1} \leq \rho$,

$$\epsilon \log \mathbb{P}\left(\left\| u^{\epsilon,u_0} - \tilde{\mathbf{S}}(u_0,f) \right\|_{X^{(T,p)}} \geq \delta; \left\| \sqrt{\epsilon}W - f \right\|_{\mathrm{C}\left([0,T];\mathrm{H}^s_{\mathbb{R}}\right)} < \gamma_{\rho,f,\delta} \right) \leq -L.$$

Let us denote by $O_{\rho,f,\delta}$ the set $O_{\rho,f,\delta} = B_{C\left([0,T];H^s_{\mathbb{R}}\right)}(f, \gamma_{\rho,f,\delta})$. The family $(O_{\rho,f,\delta})_{f\in C_a}$ is a covering by open sets of the compact set C_a, thus there exists a finite sub-covering of the form $\bigcup_{i=1}^N O_{\rho,f_i,\delta}$. We can now write

$$\begin{array}{rl}\mathbb{P}\left(u^{\epsilon,u_0} \in A^{u_0}_{\tilde{a}}\right) \leq & \mathbb{P}\left(\left\{u^{\epsilon,u_0} \in A^{u_0}_{\tilde{a}}\right\} \cap \left\{\sqrt{\epsilon}W \in \bigcup_{i=1}^N O_{\rho,f_i,\delta}\right\}\right) \\ & +\mathbb{P}\left(\sqrt{\epsilon}W \notin \bigcup_{i=1}^N O_{\rho,f_i,\delta}\right) \\ \leq & \sum_{i=1}^N \mathbb{P}\left(\left\{u^{\epsilon,u_0} \in A^{u_0}_{\tilde{a}}\right\} \cap \left\{\sqrt{\epsilon}W \in O_{\rho,f_i,\delta}\right\}\right) \\ & +\mathbb{P}\left(\sqrt{\epsilon}W \notin C_a\right) \\ \leq & \sum_{i=1}^N \mathbb{P}\left(\left\{\left\|u^{\epsilon,u_0} - \tilde{\mathbf{S}}(u_0, f)\right\|_{X^{(T,p)}} \geq \delta\right\} \cap \left\{\sqrt{\epsilon}W \in O_{\rho,f_i,\delta}\right\}\right) \\ & + \exp\left(-\frac{a}{\epsilon}\right),\end{array}$$

for $\epsilon \leq \epsilon_0$ for some ϵ_0 positive. We used that

$$d_{C([0,T];H^1)}\left(\tilde{\mathbf{S}}(u_0, f), A^{u_0}_{\tilde{a}}\right) \geq \delta,$$

which is a consequence of the definition of the sets $A^{u_0}_{\tilde{a}}$.

As a consequence, for $\epsilon \leq \epsilon_0 \wedge (\min_{i=1,..,N} \epsilon_{u_0,f_i})$ we obtain for u_0 in B^1_ρ,

$$\mathbb{P}\left(u^{\epsilon,u_0} \in A^{u_0}_{\tilde{a}}\right) \leq N \exp\left(-\frac{L}{\epsilon}\right) + \exp\left(-\frac{a}{\epsilon}\right),$$

and for ϵ_1 small enough, for every $\epsilon \in (0, \epsilon_1)$,

$$\epsilon \log \mathbb{P}\left(u^{\epsilon,u_0} \in A^{u_0}_{\tilde{a}}\right) \leq \epsilon \log 2 + (\epsilon \log N - L) \vee (-a).$$

If ϵ_1 is also chosen such that $\epsilon_1 < \frac{\gamma}{\log(2)} \wedge \frac{L-a}{\log(N)}$ we obtain

$$\epsilon \log \mathbb{P}\left(u^{\epsilon,u_0} \in A^{u_0}_{\tilde{a}}\right) \leq -\tilde{a} - \gamma,$$

which holds for every u_0 such that $\|u_0\|_{H^1} \leq \rho$.

We consider now the lower bound (ii). Take a, ρ, T and δ positive. The continuity of $\tilde{\mathbf{S}}(u_0, \cdot)$, to be proved as in [82], along with the compactness of C_a give that for u_0 such that $\|u_0\|_{H^1} \leq \rho$ and w in $K^{u_0}_T(a)$, there exists f such that $w = \tilde{\mathbf{S}}(u_0, f)$ and $I^{u_0}_T(w) = I^W_T(f)$. Take $L > I^{u_0}(w)$. Choose $\epsilon_{\rho,f,\delta}$ positive and $O_{\rho,f,\delta}$, the ball centered at f of radius $\gamma_{\rho,f,\delta}$ defined as previously, such that for every $\epsilon \leq \epsilon_{\rho,f,\delta}$ and u_0 such that $\|u_0\|_{H^1} \leq \rho$,

$$\epsilon \log \mathbb{P}\left(\left\|u^{\epsilon,u_0} - \tilde{\mathbf{S}}(u_0, f)\right\|_{X^{(T,p)}} \geq \delta; \left\|\sqrt{\epsilon}W - f\right\|_{C\left([0,T];H^s_{\mathbb{R}}\right)} < \gamma_{\rho,f,\delta}\right) \leq -L.$$

We obtain

$$\begin{aligned}\exp\left(-\frac{I_T^W(f)}{\epsilon}\right) &\le \mathbb{P}\left(\sqrt{\epsilon}W \in O_{\rho,f,\delta}\right)\\ &\le \mathbb{P}\left(\left\{\left\|u^{\epsilon,u_0} - \tilde{\mathbf{S}}(u_0,f)\right\|_{X^{(T,p)}} \ge \delta\right\} \cap \left\{\sqrt{\epsilon}W \in O_{\rho,f,\delta}\right\}\right)\\ &\quad + \mathbb{P}\left(\left\|u^{\epsilon,u_0} - \tilde{\mathbf{S}}(u_0,f)\right\|_{X^{(T,p)}} < \delta\right).\end{aligned}$$

Thus, for $\epsilon \le \epsilon_{\rho,f,\delta}$, for every u_0 such that $\|u_0\|_{\mathrm{H}^1} \le \rho$,

$$-I^{u_0}(w) \le \epsilon \log 2 + \left(\epsilon \log \mathbb{P}\left(\left\|u^{\epsilon,u_0} - \tilde{\mathbf{S}}(u_0,f)\right\|_{X^{(T,p)}} < \delta\right)\right) \vee (-L)$$

and for ϵ_1 small enough and such that $\epsilon_1 \log(2) < \gamma$, for every ϵ positive such that $\epsilon < \epsilon_1$, for every u_0 such that $\|u_0\|_{\mathrm{H}^1} \le \rho$,

$$-I^{u_0}(w) - \gamma \le \epsilon \log \mathbb{P}\left(\left\|u^{\epsilon,u_0} - \tilde{\mathbf{S}}(u_0,f)\right\|_{X^{(T,p)}} < \delta\right).$$

It ends the proof of (i) and (ii). □

Appendix E

Large deviations and support for one-dimensional NLS equations with a fractional additive noise

Abstract: In this article we consider one-dimensional stochastic nonlinear Schrödinger equations driven by a fractional additive noise. We prove the local well posedness of the Cauchy problem in H^1, sample path large deviations and a support result. The latter results are stated in a space of exploding paths.

E.1 Introduction

We shall consider in this article stochastic nonlinear Schrödinger (NLS) equations, with a Kerr nonlinearity, driven by fractional noises of additive type. With the Itô notations, the stochastic evolution equation is written

$$i\mathrm{d}u - (\Delta u + \lambda |u|^{2\sigma} u)\mathrm{d}t = \mathrm{d}W^H, \quad \lambda = \pm 1, \tag{E.1.1}$$

u is a complex valued function of time and space. The initial datum u_0 is a function of H^1. We consider weak solutions in the sense used in the analysis of partial differential equations or equivalently mild solutions which satisfy

$$u(t) = U(t)u_0 - i\lambda \int_0^t U(t-s)(|u(s)|^{2\sigma} u(s))ds - i\int_0^t U(t-s)dW^H(s), \tag{E.1.2}$$

where $(U(t))_{t\in\mathbb{R}}$ is the Schrödinger linear group on H^1 generated by the skew adjoint unbounded operator $-i\Delta$. Recall that the group is a group of isometries on the Sobolev spaces based on L^2. We consider the one-dimensional equation in H^1. Under such assumptions the nonlinearity is Lipschitz on the bounded sets. Otherwise the Strichartz inequalities are needed to treat the nonlinearity and it is required that the stochastic convolution $\int_0^t U(t-s)dW^H(s)$ satisfies certain integrability properties which seems much more difficult to prove than in the case of the standard Brownian motion treated in [37]. The well posedness of the Cauchy problem associated to (E.1.1) in the deterministic case depends on the size of σ. If $\sigma < 2$, the nonlinearity is subcritical and the Cauchy problem is globally well posed. If $\sigma = 2$, critical nonlinearity, or $\sigma > 2$, supercritical nonlinearity, the Cauchy problem is locally well posed in H^1 and solutions may blow up in finite time; see [99]. The local well posedness and results on the the global existence when the noise is derived from a Wiener process are proved in [37].

Here we have denoted by W^H a fractional Wiener process, the fractional noise is then the time derivative in the sense of distributions of the continuous in time Hilbert space valued Gaussian process. The parameter H is called the Hurst parameter; it belongs to $(0, 1)$. When $H \neq \frac{1}{2}$, the noise is colored in both time and space. The coloration in space assumption is necessary since the group does not have regularizing properties in the Sobolev spaces based on L^2. Note that in optics the time variable corresponds to space and the space variable to some retarded time. Therefore in such situations we introduce extra correlations in space. The white in space, thus in time, noise can be approximated considering the limit of a sequence of colored noises mimicking the white in space noise in the limit; see [44, 81]. Fractional noises have been introduced in hydrology, economics and telecommunications. Though mostly white noises are considered for perturbations of the NLS equations in the physics literature, fractional noises could possibly also be relevant; however it is of interest from a mathematical point of view.

We extend the results of [81] by considering more general driving noises. Note that in [82] we have studied large deviations for a noise of multiplicative type. In [44] we have applied our results to the problem of error in soliton transmission while in [84] we have studied the problem of exit from a domain of attraction.

In this article we prove that the Cauchy problem is locally well posed and we give a sample path large deviation principle (LDP) in a space of exploding paths, we also prove a support theorem. Note that the proofs hold for quite general SPDEs with a locally Lipschitz nonlinearity and a fractional additive noise as long as we may prove that the stochastic convolution is a continuous in time process.

In the article we do not investigate global well posedness. This would require the application of the Itô formula, see [3], for the Hamiltonian and mass to a certain power as in [37]. The integrands of the stochastic integrals are anticipating and integrals remain in a Skohorod sense. It seems to be much more complicated than in the standard case. It is also the reason why we did not consider the case of multiplicative noises. This will be the object of future investigations.

E.2 Preliminaries

The space of Lebesgue square integrable functions L^2 with the inner product defined by $(u,v)_{\mathrm{L}^2} = \mathfrak{Re}\int_{\mathbb{R}} u(x)\overline{v}(x)dx$ is a Hilbert space. The Sobolev spaces H^s are the Hilbert spaces of functions of L^2 with partial derivatives up to order s in L^2. When s is fractional it is defined classically via the Fourier transform. If I is an interval of $\mathbb{R}$, $(E, \|\cdot\|_E)$ a Banach space and r belongs to $[1,\infty]$, then $\mathrm{L}^r(I;E)$ is the space of strongly Lebesgue measurable functions f from I into E such that $t \to \|f(t)\|_E$ is in $\mathrm{L}^r(I)$. The integral is the Bochner integral. The space of bounded operators from B to B', two Banach spaces, is denoted by $\mathcal{L}_c(B,B')$. The space of Hilbert-Schmidt operators Φ from E to E', two Hilbert spaces, is denoted by $\mathcal{L}_2(E,E')$, when endowed with the norm $\|\Phi\|^2_{\mathcal{L}_2(E,E')} = \mathrm{tr}\Phi\Phi^* = \sum_{j\in\mathbb{N}} \|\Phi e_j\|^2_{E'}$ where $(e_j)_{j\in\mathbb{N}}$ is a complete orthonormal system of E it is a is a Hilbert space. We denote by $\mathcal{L}_2^{0,s}$ the above space when $E = \mathrm{L}^2$ and $E' = \mathrm{H}^s$.

We denote by $x\wedge y$ and by $x \vee y$ respectively the minimum and maximum of x and y. Recall that a rate function I is a lower semicontinuous function and that it is good if for every c positive, $\{x : I(x) \le c\}$ is a compact set.

Let us now recall, for the sake of completeness, the main properties of the fractional Brownian motion.

A fBm is a centered Gaussian process with stationary increments

$$\mathbb{E}\left(\left|\beta^H(t) - \beta^H(s)\right|^2\right) = |t-s|^{2H}, \quad t,s > 0.$$

The covariance is given by

$$\mathbb{E}\left(\beta^H(t)\beta^H(s)\right) = \frac{1}{2}\left(s^{2H} + t^{2H} - |s-t|^{2H}\right).$$

The increments are no longer independent for $H \neq \frac{1}{2}$. The covariance of future and past increments is negative if $H < \frac{1}{2}$ and positive if $H > \frac{1}{2}$.

FBms present long range dependence for $H > \frac{1}{2}$ as the covariance between increments at a distance u decays as u^{2H-2}. These processes are also self-similar, *i.e.* the law of the paths $t \mapsto \beta^H(at)$ where a is positive are that of $t \mapsto a^H \beta^H(t)$. Note that the solution of the NLS equation displays a self similar behavior near blow-up for supercritical nonlinearities but in the space variable, see [136]. The case where $H = \frac{1}{2}$ is that of the standard Brownian motion.

Enlarging if necessary the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the fBm may be defined in terms of a standard Brownian motion $(\beta(t))_{t\geq 0}$ via a square integrable triangular kernel K^H, *i.e.* $K^H(t,s) = 0$ if $s > t$,

$$\beta^H(t) = \int_0^t K^H(t,s) d\beta(s),$$

where

$$K^H(t,s) = c_H (t-s)^{H-\frac{1}{2}} + c_H \left(\frac{1}{2} - H\right) \int_s^t (u-s)^{H-\frac{3}{2}} \left(1 - \left(\frac{s}{u}\right)^{\frac{1}{2}-H}\right) du, \tag{E.2.1}$$

and

$$c_H = \left(\frac{2H\Gamma\left(\frac{3}{2} - H\right)}{\Gamma\left(H + \frac{1}{2}\right)\Gamma\left(2 - 2H\right)}\right)^{\frac{1}{2}}.$$

From (E.2.1) we may obtain

$$\frac{\partial K^H}{\partial t}(t,s) = c_H \left(\frac{1}{2} - H\right) (t-s)^{H-\frac{3}{2}} \left(\frac{s}{t}\right)^{\frac{1}{2}-H}. \tag{E.2.2}$$

We shall now denote by $(\mathcal{F}_t)_{t\geq 0}$ the filtration generated by the above fBm. The fBm admits a modification with $\alpha-$Hölder continuous sample paths where $\alpha < H$; see for example [42]. The paths also have $\frac{1}{H}-$finite variation. For $H \neq \frac{1}{2}$, the fBm is neither Markovian nor a semi martingale. As a consequence of the latter stochastic integration with respect to fBM requires a different approach from that of integration with respect to semi-martingales. Several approaches to it exist and we will adopt that of [3] where it is defined as a Skohorod integral. A rough paths approach may also be considered, see for example [33, 113]. FBms are particular Volterra processes, see for example [41], defined for T positive as

$$X(t) = \int_0^t K(t,s) d\beta(s), \quad K \in \mathrm{L}^2\left([0,T] \times [0,T]\right),\ T > 0,\ K(t,s) = 0 \text{ if } s > t.$$

The covariance of such processes is

$$R(t,s) = \int_0^{t\wedge s} K(t,r)K(s,r)dr,$$

the covariance operator when the process is considered as a $\mathrm{L}^2(0,T)-$random variable is nuclear, *i.e.* it has finite trace, and is defined through the kernel $R(t,s)$, *i.e.* for h in $\mathrm{L}^2(0,T)$, $Rh(t) = \int_0^T R(t,s)h(s)ds$. The operator R is such that $R = KK^*$ where K is the Hilbert-Schmidt operator that satisfies for $h \in \mathrm{L}^2(0,T)$, $Kh(t) = \int_0^T K(t,s)h(s)ds = \int_0^t K(t,s)h(s)ds$ and K^* is its adjoint. These processes admit modifications with continuous sample paths; they are Gaussian processes. Thus it is classical that the reproducing kernel Hilbert space (RKHS) of the Gaussian measure in $\mathrm{L}^2(0,T)$, equal to the range of $R^{\frac{1}{2}}$ denoted by Im $R^{\frac{1}{2}}$ with the norm of the image structure, is also equal to Im K; it is also the RKHS of the measure in $\mathrm{C}([0,T])$ which is a Banach space continuously embedded in $\mathrm{L}^2(0,T)$. When we consider directly the RKHS of the measure on $\mathrm{C}([0,T])$ we also know that it is isometric to the closure in $\mathrm{L}^2(\mu)$, where μ is the Gaussian measure, of the dual of the Banach space defined by means of the evaluation at points t in $[0,T]$ and thus to the first Wiener chaos. After such characterizations of the RKHS of the fBm it is then standard fact, see for example [8, 53], that the rate function of a LDP for the family of Gaussian measures defined as direct images of μ via the mapping $x \mapsto \sqrt{\epsilon}x$ is given by $\frac{1}{2}\|\cdot\|_{\mathcal{H}}^2$ and that the support of the law of the Gaussian measure is the closure of the RKHS for the norm of the Banach space. We aim to transport such results to the law of the solution of the stochastic NLS equation driven by such noises.

In order to define a stochastic integration we consider another RKHS which is at the level of the noise and not of the process, we denote it by $\mathcal{H}$. It may be seen as generated by step functions on $[0,T]$; the stochastic integral of a step function $\mathbb{1}_{[0,T]}$ should coincide with the evaluation at point t of the fBm. The set of such step functions is denoted by $\mathcal{E}$. The inner product is then defined as

$$R(t,s) = \left\langle \mathbb{1}_{[0,t]}, \mathbb{1}_{[0,s]}\right\rangle_{\mathcal{H}} = \left(K(t,\cdot)\mathbb{1}_{[0,t]}, K(s,\cdot)\mathbb{1}_{[0,s]}\right)_{\mathrm{L}^2(0,T)}.$$

Also a representation of $\mathcal{H}$ may be obtained considering the linear operator K_T^* from $\mathcal{E}$ into $\mathrm{L}^2(0,T)$ defined for φ in $\mathcal{E}$ by

$$\left(K_T^*\varphi\right)(s) = \varphi(s)K(T,s) + \int_s^T (\varphi(t)-\varphi(s))K(dt,s). \tag{E.2.3}$$

It is such that for any φ in $\mathcal{E}$ and h in $\mathrm{L}^2(0,T)$ we have

$$\int_0^T \left(K_T^*\varphi\right)(t)h(t)dt = \int_0^T \varphi(t)(Kh)(dt).$$

The space $\mathcal{H}$ may then be represented as the closure of $\mathcal{E}$ with respect to the norm $\|\varphi\|_{\mathcal{H}} = \|K_T^*\varphi\|_{\mathrm{L}^2(0,T)}$. The operator K_T^* is then an isometry between $\mathcal{H}$ and a closed subspace of $\mathrm{L}^2(0,T)$; we write $\mathcal{H} = \left(K_T^*\right)^{-1}\left(\mathrm{L}^2(0,T)\right)$. The above duality relation allows to extend integration with respect to $Kh(dt)$ to integrands in $\mathcal{H}$; it extends that of step functions. It also allows to define a stochastic integration and for integrands φ in $\mathcal{H}$, the Skohorod integral satisfies

$$\delta^X(\varphi) = \int_0^T \left(K_T^*\varphi\right)(t)\delta\beta(t) = \int_0^T \left(K_T^*\varphi\right)(t)d\beta(t).$$

From now on we shall restrict our attention to the particular case of the fBm and denote the kernel and the operator by K instead of K^H.

We shall use several time the following necessary property that we may easily check for the fBm thanks to (E.2.1) and (E.2.3) that for $0 < t < T$,

$$\left(K_T^*\mathbb{1}_{[0,t]}\varphi\right)(s) = \left(K_t^*\varphi\right)(s)\mathbb{1}_{[0,t]}(s). \tag{E.2.4}$$

Recall that for the smoother kernels such that $H > \frac{1}{2}$, relation (E.2.3) has the simpler and more natural form

$$\left(K_T^*\varphi\right)(s) = \int_s^T \varphi(r)K(dr,s). \tag{E.2.5}$$

The formulation in (E.2.3) allows to extend this definition to singular kernels, *i.e.* when $H < \frac{1}{2}$. For H such that $H > \frac{1}{2}$, the inner product in $\mathcal{H}$ of φ and ψ is given by

$$\begin{aligned} \langle\varphi,\psi\rangle_{\mathcal{H}} &= \int_0^T\int_0^T \varphi(u)\psi(v)\int_0^{u\wedge v}\frac{\partial K}{\partial u}(u,s)\frac{\partial K}{\partial v}(v,s)dsdudv \\ &= c_H^2\left(H-\tfrac{1}{2}\right)^2 B\left(2-2H, H-\tfrac{1}{2}\right)\int_0^T\int_0^T \varphi(u)\psi(v)|u-v|^{2H-2}dudv, \end{aligned}$$

from a computation given in [4]; B denotes the Beta function. It corresponds to the covariance of the stochastic integrals with respect to the fBm

$$\mathbb{E}\left[\int_0^T \varphi(u)d\beta^H(u)\int_0^T \psi(v)d\beta^H(v)\right];$$

the space $\mathcal{H}$ is thus what would be a RKHS at the level of the noise in $\mathrm{L}^2(0,T)$ which covariance is $\int_0^{u\wedge v}\frac{\partial K}{\partial u}(u,s)\frac{\partial K}{\partial v}(v,s)ds$.

In infinite dimensions, we assume that W^H is the direct image by a Hilbert-Schmidt operator Φ of a cylindrical fractional Wiener process on L^2, *i.e.* $W^H = \Phi W_c^H$.

We shall assume that

$$\Phi \text{ belongs to } \mathcal{L}_2(\mathrm{L}^2, \mathrm{H}^{1+\gamma}) \text{ with } 0 \leq \gamma < 1 \text{ for } H > \tfrac{1}{2} \quad \text{and } (1-2H) < \gamma < 1 \text{ for } H < \tfrac{1}{2}. \tag{A}$$

This assumption will be used along with the fact that for any $\gamma \in (0,1)$ and t a real number

$$\|U(t) - I\|_{\mathcal{L}_c(\mathrm{H}^{1+\gamma},\mathrm{H}^1)} \leq 2^{\frac{1-\gamma}{2}} |t|^{\frac{\gamma}{2}}; \tag{E.2.6}$$

it could be proved using the Fourier transform.

A cylindrical fractional Wiener process on a Hilbert space is such that for every orthonormal basis $(e_j)_{j\in\mathbb{N}}$ there exists independent fractional Brownian motions (fBm) $\left(\beta_j^H(t)\right)_{t\geq 0}$ such that $W_c(t) = \sum_{j\in\mathbb{N}} \beta_j^H(t) e_j$. Note that in what follows it is also possible to assume that the parameter H takes different values H_j on each coordinate. The Hilbert-Schmidt assumption is known to be the exact assumption needed in Hilbert space so that the marginals $W^H(t)$ are well defined Gaussian measures. Stochastic integration with respect to fractional Wiener processes in a Hilbert space E', see for example [137], when integrands are deterministic is defined as above but for step functions multiplied by elements of the Hilbert space. It is such that a scalar product by an element of E' is the one dimensional stochastic integral of the scalar product of the integrand. Operators K_T^* are still well defined when the RKHS $\mathcal{H}$ is made of functions with values in E'. Integrals of deterministic bounded operator valued integrands Λ from the Hilbert space E to E' are defined for t positive as

$$\int_0^t \Lambda(s) dW^H(s) = \sum_{j\in\mathbb{N}} \int_0^t \Lambda(s)\Phi e_j d\beta_j^H(s) = \sum_{j\in\mathbb{N}} \int_0^t \left(K_t^* \Lambda(\cdot)\Phi e_j\right)(s) d\beta_j(s),$$

when $(\Lambda(t))_{t\in[0,T]}$ is such that

$$\sum_{j\in\mathbb{N}} \int_0^T \| \left(K_T^* \Lambda(\cdot)\Phi e_j\right)(t)\|_{E'}^2 dt < \infty.$$

Note that the duality relation (E.1.2) still holds, the integral is a Bochner integral, and that K_T^* commutes with the scalar product with an element of E'. We assume now that $E = \mathrm{L}^2$, $E' = \mathrm{H}^1$ and that $(e_j)_{j\in\mathbb{N}}$ is an orthonormal basis of L^2.

We may also check from (E.1.2) that the linear group $(U(t))_{t\in\mathbb{R}}$ commutes with K_T^*.

Let us now present the space of exploding paths. Indeed, since the nonlinearity is only Lipschitz on the bounded sets of H^1, solutions may blow up in finite time. We shall proceed as in [81]; see the reference for more details. We add a point Δ to the space H^1 and embed the space with the topology such that its open sets are the open sets of H^1 and the complement in $\mathrm{H}^1\cup\{\Delta\}$ of the closed bounded sets of H^1. The set $\mathrm{C}([0,\infty);\mathrm{H}^1\cup\{\Delta\})$ is then well defined. We denote the blow-up time of f in $\mathrm{C}([0,\infty);\mathrm{H}^1\cup\{\Delta\})$ by $\mathcal{T}(f)=\inf\{t\in[0,\infty):f(t)=\Delta\}$, with the convention that $\inf\emptyset=\infty$. We may now define the set

$$\mathcal{E}(\mathrm{H}^1)=\left\{f\in\mathrm{C}([0,\infty);\mathrm{H}^1\cup\{\Delta\}):f(t_0)=\Delta\Rightarrow\forall t\geq t_0,\ f(t)=\Delta\right\},$$

it is endowed with the topology defined by the neighborhood basis

$$V_{T,r}(\varphi_1)=\left\{\varphi\in\mathcal{E}(\mathrm{H}^1):\mathcal{T}(\varphi)>T,\ \|\varphi_1-\varphi\|_{\mathrm{C}([0,T];\mathrm{H}^1)}\leq r\right\},$$

of φ_1 in $\mathcal{E}(\mathrm{H}^1)$ given $T<\mathcal{T}(\varphi_1)$ and r positive. The space is a Hausdorff topological space and thus we may consider applying the Varadhan contraction principle.

E.3 Local well posedness and necessary results

We shall proceed as in [37] though we do not have to check the integrability property. Indeed in dimension one we do not need the Strichartz inequalities. Let us first check the following lemma.

Lemma E.3.1 *The stochastic convolution $Z:\ t\mapsto\int_0^t U(t-s)dW^H(s)$ is well defined. Moreover, under assumption (A) it has a modification with $\alpha-$Hölder continuous sample paths where $\alpha<\frac{\gamma}{2}\wedge H$. If $H\geq\frac{1}{2}$ and $\gamma=0$ the modification is only continuous. It defines a $\mathrm{C}\left([0,\infty);\mathrm{H}^1\right)$ random variable. Moreover, the direct images $\mu^{Z,T}$ of its law μ^Z by the restriction on $\mathrm{C}\left([0,T];\mathrm{H}^1\right)$ for T positive are centered Gaussian measures.*

Proof. The stochastic convolution is well defined since for t positive

$$\begin{aligned}
&\textstyle\sum_{j\in\mathbb{N}}\int_0^t\|(K_t^*U(t-\cdot)\Phi e_j)(u)\|_{\mathrm{H}^1}^2du\\
&=\textstyle\sum_{j\in\mathbb{N}}\int_0^t\left\|U(-u)\Phi e_jK(t,u)+\int_u^t(U(-r)-U(-u))\Phi e_jK(dr,u)\right\|_{\mathrm{H}^1}^2du\\
&\leq 2\|\Phi\|_{\mathcal{L}_2^{0,1}}^2\int_0^tK(t,u)^2du+2\textstyle\sum_{j\in\mathbb{N}}\int_0^t\left\|\int_u^t(U(-r)-U(-u))\Phi e_jK(dr,u)\right\|_{\mathrm{H}^1}^2du\\
&\leq T_1+T_2
\end{aligned}$$

The integral in T_1 is equal to $\mathbb{E}[(\beta^H(t))^2] = t^{2H}$. To obtain an upper bound for the second term we have to distinguish the cases $H < \frac{1}{2}$ and $H > \frac{1}{2}$. When $H < \frac{1}{2}$, we apply (E.2.6) and obtain

$$T_2 \leq 2^{2-\gamma}\|\Phi\|^2_{\mathcal{L}_2^{0,1+\gamma}} \left(\frac{1}{c_H\left(H-\frac{1}{2}\right)}\right)^2 \int_0^t \left(\int_u^t (r-u)^{H-\frac{3}{2}+\frac{\gamma}{2}} \left(\frac{r}{u}\right)^{H-\frac{1}{2}} dr\right)^2 du$$

thus

$$T_2 \leq 2^{2-\gamma}\|\Phi\|^2_{\mathcal{L}_2^{0,1+\gamma}} \left(\frac{1}{c_H\left(H-\frac{1}{2}\right)}\right)^2 \int_0^t \left(\int_u^t (r-u)^{H-\frac{3}{2}+\frac{\gamma}{2}} dr\right)^2 du,$$

the integral is well defined since $H - \frac{3}{2} + \frac{\gamma}{2} > -1$. We finally obtain

$$T_2 \leq \frac{2^{2-\gamma}\|\Phi\|^2_{\mathcal{L}_2^{0,1+\gamma}}}{2H-1+\gamma} \left(\frac{1}{c_H\left(H-\frac{1}{2}\right)\left(H-\frac{1}{2}+\frac{\gamma}{2}\right)}\right)^2 t^{2H+\gamma}.$$

When $H > \frac{1}{2}$ we do not need to use (A), the kernel is null on the diagonal and its derivative is integrable. We obtain

$$T_2 \leq 2^{4-\gamma}\|\Phi\|^2_{\mathcal{L}_2^{0,1+\gamma}} \int_0^t K(t,u)^2 du = 2^{4-\gamma}\|\Phi\|^2_{\mathcal{L}_2^{0,1+\gamma}} t^{2H}.$$

The fact that $\mu^{Z,T}$ are Gaussian measures follows from the fact that Z is defined as

$$\sum_{j\in\mathbb{N}} \int_0^t \left(K_T^* \mathbb{1}_{[0,t]}(\cdot) U(t-\cdot)\Phi e_j\right)(s) d\beta_j(s).$$

The law is Gaussian since the law of the action of an element of the dual is a pointwise limit of Gaussian random variables; see for example [81].

Note that as we are dealing with a centered Gaussian process, we may control the higher moments controlling the second order moments; see [34] for a proof in the infinite dimensional setting. It is therefore enough to show that for every positive T there exists positive C and α such that for every $(t,s) \in [0,T]^2$.

$$\mathbb{E}\left[\|Z(t) - Z(s)\|^2_{\mathrm{H}^1}\right] \leq C|t-s|^\alpha,$$

and then control higher moments and conclude with the Kolmogorov criterion.

When $0 < s < t$, we have

$$\begin{aligned} Z(t) - Z(s) = \;& U(s)\left(U(t-s) - I\right) \textstyle\sum_{j\in\mathbb{N}} \int_0^T \left(K_T^* \mathbb{1}_{[0,t]}(\cdot) U(-\cdot)\Phi e_j\right)(w) d\beta_j(w) \\ & + U(s) \textstyle\sum_{j\in\mathbb{N}} \int_0^T \left(\left(K_t^* U(-\cdot)\Phi e_j\right)(w) - \left(K_s^* U(-\cdot)\Phi e_j\right)(w)\right) d\beta_j(w) \\ = \;& \tilde{T}_1(t,s) + \tilde{T}_2(t,s). \end{aligned}$$

Let us begin with the case where $\gamma > 0$. We may write

$$\mathbb{E}\left[\|Z(t)-Z(s)\|_{\mathrm{H}^1}^2\right] \leq 2\mathbb{E}\left[\left\|\tilde{T}_1(t,s)\right\|_{\mathrm{H}^1}^2\right] + 2\mathbb{E}\left[\left\|\tilde{T}_2(t,s)\right\|_{\mathrm{H}^1}^2\right].$$

From the above when (A) is such that $\gamma > 0$ there exists a constant $C(T)$ such that

$$\begin{aligned}
&\mathbb{E}\left[\left\|\tilde{T}_1(t,s)\right\|_{\mathrm{H}^1}^2\right] \\
&\leq \|U(t-s)-I\|_{\mathcal{L}_c(\mathrm{H}^{1+\gamma},\mathrm{H}^1)}^2 \sum_{j\in\mathbb{N}} \int_0^T \left\|\left(K_T^*\mathbb{1}_{[0,t]}(\cdot)U(-\cdot)\Phi e_j\right)^2(w)\right\|_{\mathrm{H}^1}^2 dw \\
&\leq 2^{1-\gamma}C(T)\|\Phi\|_{\mathcal{L}_2^{0,1}}^2|t-s|^\gamma.
\end{aligned}$$

Also,

$$\begin{aligned}
&\mathbb{E}\left[\left\|\tilde{T}_2(t,s)\right\|_{\mathrm{H}^1}^2\right] \\
&= \sum_{j\in\mathbb{N}} \int_0^T \| U(-u)\Phi e_j K(t,u) + \int_u^t \left(U(-r)-U(-u)\right)\Phi e_j K(dr,u) \\
&\qquad\qquad -\varphi(u)K(s,u) - \int_u^s \left(U(-r)-U(-u)\right)\Phi e_j K(dr,u) \quad \|_{\mathrm{H}^1}^2 du \\
&\leq \sum_{j\in\mathbb{N}} \left(\tilde{T}_{21}^j + \tilde{T}_{22}^j + \tilde{T}_{23}^j\right),
\end{aligned}$$

where, using the fact that the kernel is triangular,

$$\begin{aligned}
\tilde{T}_{21}^j &= \int_0^s \left\|U(-u)\left(K(t,u)-K(s,u)\right) + \int_s^t \left(U(-r)-U(-u)\right)\Phi e_j K(dr,u)\right\|_{\mathrm{H}^1}^2 du, \\
\tilde{T}_{22}^j &= 2\int_s^t \|U(-u)\Phi e_j K(t,u)\|_{\mathrm{H}^1}^2 du \\
\tilde{T}_{23}^j &= 2\int_s^t \left\|\int_u^t \left(U(-r)-U(-u)\right)\Phi e_j K(dr,u)\right\|_{\mathrm{H}^1}^2 du.
\end{aligned}$$

We have

$$\begin{aligned}
\tilde{T}_{21}^j &= \int_0^s \left\|\int_s^t U(-r)\Phi e_j K(dr,u)\right\|_{\mathrm{H}^1}^2 du \\
&= \|\Phi e_j\|_{\mathrm{H}^1}^2 \int_0^s \left(\int_s^t |K(dr,u)|\right)^2 du \\
&= \|\Phi e_j\|_{\mathrm{H}^1}^2 \int_0^s \left(K(t,u)-K(s,u)\right)^2 du
\end{aligned}$$

thus

$$\begin{aligned}
\tilde{T}_{21}^j &\leq \|\Phi e_j\|_{\mathrm{H}^1}^2 \int_0^t \left(K(t,u)-K(s,u)\right)^2 du \\
&\leq \|\Phi e_j\|_{\mathrm{H}^1}^2\, \mathbb{E}\left[\left(\beta^H(t)-\beta^H(s)\right)^2\right] \\
&\leq \|\Phi e_j\|_{\mathrm{H}^1}^2\, |t-s|^{2H},
\end{aligned}$$

and

$$\begin{aligned}
\tilde{T}_{22}^j &= 2\left\|\Phi e_j\right\|_{\mathrm{H}^1}^2 \int_s^t K(t,u)^2 du \\
&= 2\left\|\Phi e_j\right\|_{\mathrm{H}^1}^2 \int_s^t \left(K(t,u)-K(s,u)\right)^2 du
\end{aligned}$$

thus

$$\begin{array}{ll}\tilde{T}_{22}^{j} & \leq 2\left\|\Phi e_j\right\|_{\mathrm{H}^1}^2 \int_0^t \left(K(t,u)-K(s,u)\right)^2 du \\ & \leq 2\left\|\Phi e_j\right\|_{\mathrm{H}^1}^2 |t-s|^{2H},\end{array}$$

finally the same computations as above shows that when $H<\frac{1}{2}$, since $H-\frac{3}{2}+\frac{\gamma}{2}>0$ we have

$$\begin{array}{ll}\tilde{T}_{23}^{j} & \leq 2^{2-\gamma}\|\Phi\|_{\mathcal{L}_2^{0,1+\gamma}}^2 \left(\frac{1}{c_H\left(H-\frac{1}{2}\right)}\right)^2 \int_s^t \left(\int_u^t (r-u)^{H-\frac{3}{2}+\frac{\gamma}{2}} \left(\frac{r}{u}\right)^{H-\frac{1}{2}} dr\right)^2 du \\ & \leq 2^{4-\gamma}\|\Phi\|_{\mathcal{L}_2^{0,1+\gamma}}^2 |t-s|^{2H+\gamma}.\end{array}$$

and when $H>\frac{1}{2}$ the kernel is null on the diagonal and its derivative is negative and integrable thus we have

$$\begin{array}{ll}\tilde{T}_{23}^{j} & \leq 4\int_s^t \left\|\Phi e_j\right\|_{\mathrm{H}^1}^2 \left(\int_u^t |K(dr,u)|\right)^2 du \\ & \leq 4\left\|\Phi e_j\right\|_{\mathrm{H}^1}^2 \int_s^t K(t,u)^2 du \\ & \leq 4\left\|\Phi e_j\right\|_{\mathrm{H}^1}^2 \mathbb{E}\left[|\beta^H(t)-\beta^H(s)|^2\right] \\ & \leq 4\left\|\Phi e_j\right\|_{\mathrm{H}^1}^2 |t-s|^{2H}.\end{array}$$

We may conduct similar computations when $0<t<s<T$ and we are now able to conclude that Z admits a modification with α-Hölder continuous sample paths with $\alpha<\frac{\gamma}{2}\wedge H$.

Let us now consider the case where $H>\frac{1}{2}$ and $\gamma=0$. Since the group is an isometry we have

$$\begin{array}{ll}\|Z(t)-Z(s)\|_{\mathrm{H}^1} \leq & \left\|(U(t-s)-I)\sum_{j\in\mathbb{N}}\int_0^T \left(K_T^*\mathbb{1}_{[0,t]}(\cdot)U(-\cdot)\Phi e_j\right)(w)d\beta_j(w)\right\|_{\mathrm{H}^1} \\ & +\left\|\tilde{T}_2(t,s)\right\|_{\mathrm{H}^1}.\end{array}$$

Since the group is strongly continuous and since, from the above,

$$\sum_{j\in\mathbb{N}}\int_0^T \left(K_T^*\mathbb{1}_{[0,t]}(\cdot)U(-\cdot)\Phi e_j\right)(w)d\beta_j(w)$$

belongs to H^1 the first term of the right hand side goes to zero as s converges to t. Also we may write

$$\left\|\tilde{T}_2(t,s)\right\|_{\mathrm{H}^1} \leq \|Y(t)-Y(s)\|_{\mathrm{H}^1}$$

where $(Y(t))_{t\in[0,T]}$ defined for $t\in[0,T]$ by

$$Y(t)=\sum_{j\in\mathbb{N}}\int_0^T \left(K_t^* U(-\cdot)\Phi e_j\right)(w)d\beta_j(w)$$

is a Gaussian process. We again conclude from the Kolmogorov criterion that $Y(t)$ admits a modification with continuous sample paths. Thus for such a modification of Y, Z has continuous sample paths. We shall now consider this modification.

It is now a standard fact to prove that the process defines a $\mathrm{C}\left([0,\infty);\mathrm{H}^1\right)$ random variable, see for example [81]. $\square$

In the following we consider such a modification. Let us now denote by $v^{u_0}(z)$ the solution of

$$\begin{cases} i\frac{\mathrm{d}v}{\mathrm{d}t} = \Delta v + \lambda|v-iz|^{2\sigma}(v-iz) \\ u(0) = u_0 \in \mathrm{H}^1 \end{cases},$$

where z belongs to $\mathrm{C}\left([0,\infty);\mathrm{H}^1\right)$. The local well posedness is obtained by a fixed point argument on $\mathrm{C}\left([0,T];\mathrm{H}^1\right)$ for a sufficiently small time interval; it uses the fact that the nonlinearity is Lipschitz on the bounded sets of H^1.

We then define $\mathcal{G}^{u_0}$ the mapping

$$\mathcal{G}^{u_0}: \ z \mapsto v^{u_0}(z) - iz,$$

it is such that $u^{\epsilon,u_0} = \mathcal{G}^{u_0}(\sqrt{\epsilon}Z)$ where Z is the stochastic convolution defined by $Z(t) = \int_0^t U(t-s)dW^H(s)$.

We may now check from similar arguments as in [37, 81] the two next results.

Lemma E.3.2 *The mapping*

$$\begin{array}{rcl} \mathrm{C}\left([0,\infty);\mathrm{H}^1\right) & \rightarrow & \mathcal{E}\left(\mathrm{H}^1\right) \\ z & \mapsto & \mathcal{G}^{u_0}(z) \end{array}$$

is continuous.

Theorem E.3.3 *Assume (A) and that the initial datum u_0 is $\mathcal{F}_0$ measurable with values in H^1; then there exists a unique solution to* (E.1.2) *with continuous H^1 valued paths such that $u(0) = u_0$. The solution is defined on a random interval $[0,\tau^*(u_0,\omega))$ where $\tau^*(u_0,\omega)$ is a stopping time such that*

$$\tau^*(u_0,\omega) = \infty \text{ or } \lim_{t\to\tau^*(u_0,\omega)} \|u(t)\|_{\mathrm{H}^1} = \infty.$$

Furthermore, τ^* is almost surely lower semi continuous with respect to u_0. The solution u defines a $\mathcal{E}\left(\mathrm{H}^1\right)$ random variable.

E.4 The main results

Let us first study large deviations for the laws $\mu^{u^{\epsilon,u_0}}$ on $\mathcal{E}(\mathrm{H}^1)$ of the mild solutions u^{ϵ,u_0} of

$$\left\{ \begin{array}{l} i\mathrm{d}u - (\Delta u + \lambda|u|^{2\sigma}u)\mathrm{d}t = \sqrt{\epsilon}\mathrm{d}W^H, \\ u(0) = u_0 \ \in \mathrm{H}^1. \end{array} \right. \tag{E.4.1}$$

Lemma E.4.1 *The covariance operator of Z on $\mathrm{L}^2\left(0,T;\mathrm{L}^2\right)$ is given for h in $\mathrm{L}^2\left(0,T;\mathrm{L}^2\right)$ by*

$$\begin{array}{rl} \mathcal{Q}h(t) = \sum_{j\in\mathbb{N}} \int_0^T \int_0^{t\wedge u} & \left(K_T^*\mathbb{1}_{[0,t]}(\cdot)U(t-\cdot)\Phi e_j\right)(s) \\ & \left(\left(K_T^*\mathbb{1}_{[0,u]}(\cdot)U(u-\cdot)\Phi e_j\right)(s), h(u)\right)_{\mathrm{L}^2} dsdu, \end{array}$$

when $H > \frac{1}{2}$ we may write $\mathcal{Q}h(t)$ as

$$c_H^2\left(H-\frac{1}{2}\right)^2\beta\left(2-2H, H-\frac{1}{2}\right)\int_0^T\int_0^t\int_0^s |u-v|^{2H-2}U(t-v)\Phi\Phi^*U(u-s)h(s)dudvds.$$

Also, the RKHS of $\mu^{Z,T}$ is $\mathrm{Im}\ \mathcal{Q}^{\frac{1}{2}}$ with the norm of the image structure. It is also $\mathrm{Im}\mathcal{L}$ where $\mathcal{L}$ is defined for h in $\mathrm{L}^2\left(0,T;\mathrm{L}^2\right)$ by

$$\mathcal{L}h(t) = \sum_{j\in\mathbb{N}}\int_0^t \left(K_T^*\mathbb{1}_{[0,t]}(\cdot)U(t-\cdot)\Phi e_j\right)(s)(h(s),e_j)_{\mathrm{L}^2}ds.$$

Moreover the direct image measures for ϵ positive of $x \mapsto \sqrt{\epsilon}x$ on $\mathrm{C}\left([0,\infty);\mathrm{H}^1\right)$ satisfy a LDP of speed ϵ and good rate function

$$I^Z(f) = \frac{1}{2}\inf_{h\in\mathrm{L}^2(0,\infty;\mathrm{L}^2):\mathcal{L}(h)=f}\left\{\|h\|^2_{\mathrm{L}^2(0,\infty;\mathrm{L}^2)}\right\}.$$

Proof. We may first check with the same computations as those used in Lemma E.3.1 that $\mathcal{L}$ is well defined and that for h in $\mathrm{L}^2(0,T;\mathrm{L}^2)$, $\mathcal{L}h$ belongs to $\mathrm{L}^2(0,T;\mathrm{L}^2)$; it also holds replacing L^2 by H^1. Take h and k in $\mathrm{L}^2(0,T;\mathrm{L}^2)$, we have

$$\begin{array}{l} \mathbb{E}\left[\int_0^T (Z(u),h(u))_{\mathrm{L}^2}du\int_0^T (Z(t),k(t))_{\mathrm{L}^2}dt\right] \\ = \sum_{j\in\mathbb{N}}\mathbb{E}\left[\int_0^T\int_0^T\left(\int_0^T\left(K_T^*\mathbb{1}_{[0,u]}(\cdot)U(u-\cdot)\Phi e_j\right)(s)d\beta_j(s),h(u)\right)_{\mathrm{L}^2}\right. \\ \qquad\qquad\qquad \left.\left(\int_0^T\left(K_T^*\mathbb{1}_{[0,t]}(\cdot)U(t-\cdot)\Phi e_j\right)(v)d\beta_j(v),k(t)\right)_{\mathrm{L}^2}\right] \\ = \int_0^T (\mathcal{Q}h(t),k(t))_{\mathrm{L}^2}dt \end{array}$$

where $\mathcal{Q}$ is defined in the lemma. The result for $H > \frac{1}{2}$ is obtained with the particular form of the inner product in $\mathcal{H}$ for such values of H.

Checking that for k in $\mathrm{L}^2(0,T;\mathrm{L}^2)$,

$$\mathcal{L}^* k(s) = \sum_{j\in\mathbb{N}} \int_s^T \left(\left(K_T^* \mathbb{1}_{[0,t]}(\cdot) U(t-\cdot)\Phi e_j\right)(s), k(t)\right)_{\mathrm{L}^2} e_j dt,$$

we obtain that $\mathcal{Q} = \mathcal{L}\mathcal{L}^*$.

We may thus deduce, see for example [81], that the RKHS of $\mu^{Z,T}$ is also $\mathrm{Im}\mathcal{L}$ with the norm of the image structure. It is indeed the RKHS of the direct image of $\mu^{Z,T}$ on $\mathrm{L}^2(0,T;\mathrm{L}^2)$ but it is standard fact, see for example [34, 81], that the two measures have same RKHS.

We may now deduce from the general LDP for Gaussian measures on Banach spaces, see [53], that the direct images of $\mu^{Z,T}$ by the mapping $x \mapsto \sqrt{\epsilon}x$ satisfy a LDP of speed ϵ and good rate function

$$I^{Z,T}(f) = \frac{1}{2} \inf\left\{\|h\|^2_{\mathrm{Im}\mathcal{L}} : f = \mathcal{L}h\right\}$$

with the convention that $\inf\emptyset = \infty$.

We conclude letting T go to infinity using Dawson-Gartner's theorem for projective limits and Lebesgue's dominated convergence theorem. □

We may now deduce from Lemma E.3.2 and E.4.1, the fact that

$$\left(\mathcal{G}^{u_0} \circ \mathcal{L}\right)(\cdot) = \mathbf{S}(u_0, \cdot),$$

and the Varadhan contraction principle the following theorem. Recall that the Varadhan contraction principle requires that $\mathrm{C}\left([0,\infty);\mathrm{H}^1\right)$ and $\mathcal{E}(\mathrm{H}^1)$ be Hausdorff topological spaces which is true.

Theorem E.4.2 *The laws $\mu^{u^{\epsilon,u_0}}$ on $\mathcal{E}(\mathrm{H}^1)$ satisfy a LDP of speed ϵ and good rate function*

$$I^{u_0}(w) = \frac{1}{2} \inf_{h\in\mathrm{L}^2(0,\infty;\mathrm{L}^2):\mathbf{S}(u_0,h)=w} \left\{\|h\|^2_{\mathrm{L}^2(0,\infty;\mathrm{L}^2)}\right\},$$

where $\mathbf{S}(u_0,h)$ denotes the mild solution in $\mathcal{E}(\mathrm{H}^1)$ of the following control problem

$$\begin{cases} i\frac{\partial u}{\partial t} - (\Delta u + \lambda|u|^{2\sigma}u) = \Phi\dot{K}h, \\ u(0) = u_0 \ \in \mathrm{H}^1 \\ h \in \mathrm{L}^2\left(0,\infty;\mathrm{L}^2\right); \end{cases}$$

it is called the skeleton. Only the integral, or the integral in the mild formulation, of the right hand side is defined; it is by means of the duality relation.

Remark E.4.3 *We could also prove a uniform LDP.*

The characterization of the support follows with the same arguments as in [81]. However we shall recall the proof for the sake of completeness.

Theorem E.4.4 *The support of the law $\mu^{u^{1,u_0}}$ on $\mathcal{E}(\mathrm{H}^1)$ is given by*

$$\textit{supp}\ \mu^{u^{1,u_0}} = \overline{\mathrm{Im}\mathcal{L}}^{\mathcal{E}(\mathrm{H}^1)}.$$

Proof. Step 1: We have shown that $\mu^{Z;(T,p)}$ is a Gaussian measure on a Banach space and that its RKHS is $\mathrm{Im}\mathcal{L}$. Consequently, see [8] Theorem (IX,2;1), its support is $\overline{\mathrm{Im}\mathcal{L}}^{\mathrm{C}(0,T;\mathrm{H}^1)}$. Also, from the definition of the image measure we have that

$$\mu^Z\left(p_T^{-1}\left(\overline{\mathrm{Im}\mathcal{L}}^{\mathrm{C}\left([0,T];\mathrm{H}^1\right)}\right)\right) = \mu^{Z,T}\left(\overline{\mathrm{Im}\mathcal{L}}^{\mathrm{C}\left([0,T];\mathrm{H}^1\right)}\right) = 1,$$

where p_T denotes the projection of $\mathrm{C}\left([0,\infty);\mathrm{H}^1\right)$ into $\mathrm{C}\left([0,T];\mathrm{H}^1\right)$. As a consequence it follows that

$$\textit{supp}\ \mu^Z \subset \bigcap_T p_T^{-1}\left(\overline{\mathrm{Im}\mathcal{L}}^{\mathrm{C}\left([0,T];\mathrm{H}^1\right)}\right) = \overline{\mathrm{Im}\mathcal{L}}^{\mathrm{C}\left([0,\infty);\mathrm{H}^1\right)}.$$

It then suffices to show that $\mathrm{Im}\mathcal{L} \subset \textit{supp}\ \mu^Z$. Suppose that $x \notin \textit{supp}\ \mu^Z$, then there exists a neighborhood V of x in $\mathrm{C}\left([0,\infty);\mathrm{H}^1\right)$ which is a neighborhood of x in $\mathrm{C}\left([0,T];\mathrm{H}^1\right)$ for T large such that $\mu^Z(V) = 0$. Since the support of $\mu^{Z,T}$ is the closure of $\mathrm{Im}\mathcal{L}$ for the topology of $\mathrm{C}\left([0,T];\mathrm{H}^1\right)$, $V \cap \mathrm{Im}\mathcal{L} = \emptyset$ and $x \notin \mathrm{Im}\mathcal{L}$.

Step 2: We conclude using the continuity of $\mathcal{G}$.

Since $\mathcal{G}^{u_0}(\mathrm{Im}\mathcal{L}) \subset \overline{\mathcal{G}^{u_0}(\mathrm{Im}\mathcal{L})}^{\mathcal{E}(\mathrm{H}^1)}$, $\mathrm{Im}\mathcal{L} \subset (\mathcal{G}^{u_0})^{-1}\left(\overline{\mathcal{G}^{u_0}(\mathrm{Im}\mathcal{L})}^{\mathcal{E}(\mathrm{H}^1)}\right)$.

Because $\mathcal{G}^{u_0}$ is continuous, the right side is a closed set of $\mathrm{C}\left([0,\infty);\mathrm{H}^1\right)$ and from step 1,

$$\textit{supp}\ \mu^Z \subset (\mathcal{G}^{u_0})^{-1}\left(\overline{\mathrm{Im}\left(\mathcal{G}^{u_0}\circ\mathcal{L}\right)}^{\mathcal{E}(\mathrm{H}^1)}\right),$$

and

$$\mu^Z\left((\mathcal{G}^{u_0})^{-1}\left(\overline{\mathrm{Im}\mathbf{S}(u_0)}^{\mathcal{E}(\mathrm{H}^1)}\right)\right) = 1,$$

thus

$$\textit{supp}\ \mu^u \subset \overline{\mathrm{Im}\mathbf{S}(u_0)}^{\mathcal{E}(\mathrm{H}^1)}.$$

Suppose that $x \notin \textit{supp}\ \mu^{u^{1,u_0}}$, there exists a neighborhood V of x in $\mathcal{E}(\mathrm{H}^1)$ such that $\mu^{u^{1,u_0}}(V) = \mu^Z\left((\mathcal{G}^{u_0})^{-1}(V)\right) = 0$, consequently $(\mathcal{G}^{u_0})^{-1}(V)\bigcap\mathrm{Im}\mathcal{L}$ is empty and $x \notin \mathrm{Im}\mathbf{S}(u_0)$. This gives the reverse inclusion. $\square$

Bibliography

[1] F.K.H. Abdullaev, J.C. Bronski, G. Papanicolaou, Soliton perturbations and the random Kepler problem, *Physica D* **135** (2000) 369-386.

[2] F.K.H. Abdullaev, S.A. Darmanyan, F. Lederer, Evolution of randomly modulated solitons in optical fibers, *Opt. Commun.* **126** (1996) 89-94.

[3] E. Alos, O. Mazet, D. Nualart, Stochastic calculus with respect to Gaussian processes, *Ann. Probab.* **29** (1999) 766-801.

[4] E. Alos, D. Nualart, Stochastic integration with respect to the fractional Brownian motion, *Stochastics Stochastics Rep.* **75** (2003) 129-152.

[5] R. Azencott, Grandes déviations et applications, in École d'été de Probabilité de Saint-Flour, edited by P.L. Hennequin. Spinger-Verlag, Berlin, *Lecture Notes in Math.* **774** (1980) 1-176.

[6] R. Azencott, Formule de Taylor stochastique et développements asymptotiques d'intégrales de Feynman, in Séminaire de probabilités XIV,, edited by Azéma, Meyer and Yor. Spinger-Verlag, Berlin, *Lecture Notes in Math.* **921** (1982) 237-284.

[7] R. Azencott, Petites perturbations aléatoires des systèmes dynamiques: développements asymptotiques, *Bull. Sc. Math.* **109** 253-308.

[8] A. Badrikian, S. Chevet, Mesures Cylindriques, Espaces de Wiener et Fonctions Aléatoires Gaussiennes. Springer-Verlag, Berlin, *Lecture Notes in Math.* **379** (1974).

[9] F. Balkovsky, G. Falkovich, Two complementary descriptions of intermitency, *Phys. Rev. E* **57** (1998) R1231-R1234.

[10] O. Bang, P.L. Christiansen and F. If, K.Ø. Rasmussen, Temperature effects in a nonlinear model of monolayer Scheibe aggregates, *Phys. Rev. E* **49** (1994) 4627-4636.

[11] O. Bang, P.L. Christiansen, F. If, K.Ø. Rasmussen, White Noise in the Two-dimensional Nonlinear Schrödinger Equation, *Appl. Anal.* **57** (1995) 3-15.

[12] O. Bang, P.L. Christiansen, Y.B. Gaididei, K.Ø. Rasmussen, Nonlinear and stochastic modelling of energy transfer in Scheibe aggregates, *Math. and Computers in Simulation* **40** (1996) 339-358.

[13] M. Barton-Smith, A. Debussche, L. Di Menza, Numerical study of two dimensional stochastic NLS equations, *Numer. Meth. PDEs* **21** (2005) 810-842.

[14] G. Ben Arous, Méthodes de Laplace et de la phase stationnaire sur l'espace de Wiener, *Stochastics Stochastics Rep.* **25** (1988) 125-153.

[15] Y.M. Berezansky, Z.G. Sheftel, G.F. Us, *Functional Analysis*, Vol. 1. Birkhäuser Verlag, *Operator Theory Advances and Applications* **85** (1997) 125-134.

[16] C. Brunhuber, F.G. Mertens, Y. Gaididei, Thermal diffusion of envelope solitons on anharmonic atomic chains, *Eur. Phys. J. B.* **42** (2004) 103-112.

[17] Z. Brzezniak, Stochastic Partial differential equations in M-type 2 Banach spaces, *Potential Analysis* **4** (1995) 1–45.

[18] Z. Brzezniak, On stochastic convolution in Banach spaces, and applications, *Stochastics Stochastics Rep.* **61** (1997) 245-295.

[19] V.S. Buslaev, G.S. Perel'man, On the stability of solitary waves for nonlinear Schrödinger equations, *Am. Math. Soc. Transl. Ser.* **2** (1995), 74-98.

[20] V.S. Buslaev, C. Sulem, On asymptotic stability of solitary waves for nonlinear Schrödinger equations, *Ann. Int. H. Poincaré An. non Lin.* **20** (2003), 419-475.

[21] G. Buttazzo, M. Giaquinta, S. Hildebrandt, *One-dimensional Variational Problems*. Oxford University Press, Oxford, *Oxford Lecture Series in Math. and its Appl.* **15** (1998).

[22] C. Cardon-Weber, Large deviations for Burger's type SPDE, *Stochastic Process. Appl.* **84** (1999) 53-70.

[23] R. Carmona, Interest rate models: from parametric statistics to infinite dimensional stochastic analysis, lecture notes of a course at Princeton university (2001).

[24] P. Carmona, L. Coutin, G. Montseny, Stochastic integration with respect to fractional Brownian motion, *Ann. Inst. H. Poincaré Probab. Statist.* **39** (2003) 27-68.

[25] T. Cazenave, *An Introduction to Nonlinear Schrödinger Equations*. Instituto de Matématica-UFRJ Rio de Janeiro, Brazil, *Textos de Métodos Matématicos* **26** (1993).

[26] T. Cazenave, P.L. Lions, Orbital stability of satnding waves of nonlinear Schrödinger equations, *Ann. Int. H. Poincaré An. non Lin.* **85** (1982) 549-561.

[27] S. Cerrai, M. Röckner, Large deviations for stochastic reaction-diffusion with multiplicative noise and non-lipschitz reaction term, *Ann. Probab.* **32** (2004) 1100-1139.

[28] S. Cerrai, M. Röckner, Large deviations for invariant measures of stochastic reaction-diffusion systems with multiplicative noise and non-lipschitz reaction term, *Ann. Inst. H. Poincaré Probab. Statist.* **41** (2005) 69-105.

[29] F. Chenal, A. Millet, Uniform large deviations for parabolic SPDEs and applications, *Stochastic Process. Appl.* **72** (1997) 161-187.

[30] T. Chiyonobu, S. Kusuoka, The large deviation principle for hypermixing processes, *Probab. Theory Relat. Fields* **78** (1988) 627-649.

[31] S. Cuccagna, Stabilization of solutions to nonlinear Schrödinger equations, *Comm. Pure Appl. Math.* **54** (2001) 110-1145.

[32] S. Cuccagna, On asymptotic stability of ground states of NLS, *Rend.Istit. Mat. Univ. Trieste* **32** (2002) 105-118.

[33] L. Coutin, Z. Qian, Stochastic analysis, rough path analysis and fractional Brownian motions, *Probab. Theory Relat. Fields* **122** (2002) 108-140.

[34] G. Da Prato, J. Zabczyk, *Stochastic Equations in Infinite Dimensions*. Cambridge University Press: Cambridge, England, *Encyclopedia of Mathematics and its Applications* (1992).

[35] D. Dawson, G. Papanicolaou, A random wave process, *Appl. Math. Optim.* **12** (1984), 97-114.

[36] A. de Bouard, A. Debussche, A Stochastic Nonlinear Schrödinger Equation with Multiplicative Noise, *Commun. Math. Phys.* **205** (1999) 161-181.

[37] A. de Bouard, A. Debussche, The Stochastic Nonlinear Schrödinger Equation in H^1, *Stochastic Anal. Appl.* **21** (2003) 97-126.

[38] A. de Bouard, A. Debussche, On the effect of a noise on the solutions of the focusing supercritical nonlinear Schrödinger equation, *Probab. Theory Relat. Fields* **123** (2002) 76-96.

[39] A. de Bouard and A. Debussche, Blow-up for the stochastic nonlinear Schrödinger equations with multiplicative noise, *Ann. Probab.* **33** (2005) 1078-1110.

[40] A. de Bouard, A. Debussche, Finite time blow-up in the additive supercritical nonlinear Schrödinger equation: the real noise case, *Contemporary Mathematics* **301** (2002) 183-194.

[41] L. Decreusefond, Stochastic integration with respect to Volterra processes, *Ann. Inst. H. Poincaré Probab. Statist.* **41** (2005) 123-149.

[42] L. Decreusfond, A. Üstunel, Stochastic analysis of the fractional Brownian motion, *Potential Anal.* **10** (1997) 177-214.

[43] A. Debussche, L. Di Menza, Numerical simulation of focusing stochastic nonlinear Schrödinger equations, *Physica D* **162** (2002) 131-154.

[44] A. Debussche, E. Gautier, Small noise asymptotic of the timing jitter in soliton transmission, preprint IRMAR and CREST (2005).

[45] A. Debussche, C. Odasso, Ergodicity for the weakly damped stochastic non-linear Schrödinger equations, to appear in J. Evol. Equations.

[46] A. Debussche, J. Printems, Numerical simulation of the stochastic Korteweg-de Vries equation, *Physica D* **134** (1999) 200-226.

[47] P. Del Moral, J. Garnier, Genealogical particle analysis of rare events, *Ann. Appl. Probab.* **15** (2005).

[48] A. Dembo, O. Zeitouni, *Large deviation techniques and applications* (2-nd edition). Springer-Verlag, New York, *Appl. of Math.* **38** (1998).

[49] S.A. Derevyanko, S.K. Turitsyn, D.A. Yakusev, Non-gaussian statistics of an optical soliton in the presence of amplified spontaneaous emission, *Optics Letters* **28** (2003) 2097-2099.

[50] S.A. Derevyanko, S.K. Turitsyn, D.A. Yakushev, Non-Gaussian statistics of an optical soliton in the presence of amplified spontaneous emission, *Opt. Lett.* **28** (2003) 2097-2099.

[51] S.A. Derevyanko, S.K. Turitsyn, D.A. Yakushev, Fokker-Planck equation approach to the description of soliton statistics in optical fiber transmission systems, *J. Opt. Soc. Am. B* **28** (2003) 2097-2099.

[52] E. Desurvire, *Erbium-doped fiber amplifiers: principles and applications* (2-nd edition). John Wiley & Sons (2002).

[53] J.D. Deuschel, D.W. Stroock, *Large Deviations*. Academic Press, New-York, *Pure and Appl. Math.* (1986).

[54] M.D. Donsker, S.R.S. Varadhan, Asymptotic evaluation of certain Markov processes for large time, I, *Comm. Pure Appl. Math.* **28** (1975) 1-47.

[55] M.D. Donsker, S.R.S. Varadhan, Asymptotic evaluation of certain Markov processes for large time, II, *Comm. Pure Appl. Math.* **28** (1975) 279-301.

[56] M.D. Donsker, S.R.S. Varadhan, Asymptotic evaluation of certain Markov processes for large time, III, *Comm. Pure Appl. Math.* **29** (1976) 389-461.

[57] M.D. Donsker, S.R.S. Varadhan, Asymptotic evaluation of certain Markov processes for large time, IV, *Comm. Pure Appl. Math.* **36** (1983) 183-212.

[58] M.D. Donsker, S.R.S. Varadhan, Asymptotics for the Polaron, *Comm. Pure Appl. Math.* **36** (1983) 505-528.

[59] P.D. Drummond, J.F. Corney, Quantum noise in optical fibers. II. Raman jitter in soliton communications, *J. Opt. Soc. Am. B* **18** (2001) 153-161.

[60] T.E. Duncan, B. Maslowski, B. Pasik-Duncan, Fractional Brownian motion and linear stochastic equations in Hilbert space, *Stochastic Dyn.* **2** (2002) 225-250.

[61] L.C. Evans, *Partial Differential Equations*. American Mathematical Society, Providence, Rhode Island, *Grad. Stud. in Math.* **119** (1998).

[62] G.E. Falkovich, I. Kolokolov, V. Lebedev, S.K. Turitsyn, Statistics of soliton-bearing systems with additive noise, *Phys. Rev. E* **63** (2001) 025601(R).

[63] G. Falkovich, I. Kolokolov, V. Lebedev, V. Mezentsev, S.K. Turitsyn, Non-Gaussian error probability in optical soliton transmission, *Physica D* **195** (2004) 1-28.

[64] G.E. Falkovich, M.G. Stepanov, S.K. Turitsyn, Statistics of interacting optical solitons, *Phys. Rev. E* **64** (2001) 067602.

[65] G.E. Falkovich, M.G. Stepanov, Role of interaction in causing errors in optical soliton transmission, *Opt. Lett.* **27** (2003) 13-15.

[66] S. Fang, T. Zhang, A study of a class of stochastic differential equations with non-Lipschitzian coefficients, *Probab. Theory Related Fields* **132** (2005) 356-390.

[67] W.G. Farris, G. Jona-Lasinio, Large fluctuations for a nonlinear heat equation with noise, *J. Phys. A: Math. Gen.* **15** (1982) 3025-3055.

[68] M.I. Freidlin, Random perturbations of reaction-diffusion equations: the quasi-deterministic approximation, *Trans. Amer. Math. Soc.* **305** (1988) 665–697.

[69] M.I. Freidlin, M. Weber, Random perturbations of nonlinear oscillators, *Ann. Probab.* **26, 3** (1998) 925–967.

[70] M.I. Freidlin, M. Weber, A remark on random perturbations of the nonlinear pendulum, *Ann. Appl. Probab.* **9, 3** (1999) 611–628.

[71] M.I. Freidlin, M. Weber, On random perturbations of Hamiltonian systems with many degrees of freedom, *Stochastic Process. Appl.* **94** (2001) 199–239.

[72] M.I. Freidlin, M. Weber, Random perturbations of dynamical systems and diffusion processes with conservation laws, *Probab. Theory Related Fields* **128** (2004) 441–466.

[73] M.I. Freidlin, A.D. Wentzell, *Random perturbations of dynamical systems* (2-nd edition), Springer, Berlin, 1998.

[74] M.I. Freidlin, A.D. Wentzell, Diffusion processes on graphs and the averaging principle, *Ann. Probab.* **21, 4** (1993) 2215-2245.

[75] J. Fröhlich, S. Gustafson, B.L.G. Jonsson, I.M Sigal, Solitary Wave Dynamics in an external potential, *Comm. Math. Phys.* **250** (2004) 613-642.

[76] J. Garnier, Asymptotic transmission of solitons through random media, *SIAM J. Appl. Math.* **58** (1998) 1969-1995.

[77] J. Garnier, Soliton in random media with long-range correlation, *Waves Random Media* **11** (2001) 149-162.

[78] J. Garnier, R. Marty, Effective pulse dynamics in optical fibers with polarization mode dispersion, preprint LSP, Toulouse (2004).

[79] E. Gautier, EDP stochastiques : approches de type mesure martingale et semi-groupe d'évolution autour d'une équation de la neurophysiologie, mémoire du cours de lecture du DEA modélisation stochastique et statistique d'Orsay (2001).

[80] E. Gautier, Théorèmes de support dans le cas d'une diffusion et de l'équation de Burgers stochastique, mémoire du DEA modélisation stochastique et statistique d'Orsay (2001).

[81] E. Gautier, Large deviations and support results for nonlinear Schrödinger equations with additive noise and applications, *ESAIM: Probability and Statistics* **9** (2005) 74-97.

[82] E. Gautier, Uniform large deviations for the nonlinear Schrödinger equation with multiplicative noise, *Stochastic Process. Appl.* **115** (2005) 1904-1927.

[83] E. Gautier, Large deviations for stochastic nonlinear Schrödinger equations and applications, *Actes du congrès de la SMAI à Evian* (2005).

[84] E. Gautier, Exit from a neighborhood of zero for weakly damped stochastic nonlinear Schrödinger equations, preprint IRMAR and CREST (2005).

[85] T. Georges, Study of the non-Gaussian timing jitter statistics induced by soliton interactions and filtering, *Opt. Commun.* **123** (1996) 617-623.

[86] J.M. Ghidaglia, Finite dimensional behavior for weakly damped driven Schrödinger equations, *Ann. Inst. H. Poincaré An. non lin.* **5** (1988) 365-405.

[87] R.T. Glassey, On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations, *J. Math. Phys.* **18** (1977) 1794-1797.

[88] J.P. Gordon, H.A. Haus, Random walk of coherently amplified solitons in optical fiber transmission, *Opt. Lett.* **11** (1986) 665-667.

[89] J.P. Gordon, Dispersive perturbations of solitons of the nonlinear Schrödinger equation, *J. Opt. Soc. Am. B* **9** (1992) 91-97.

[90] O. Goubet, Regularity of the attractor for a weakly damped nonlinear Schrödinegr equation, *Applicable Analysis* **60** (1996) 99-119.

[91] M. Grillakis, J. Shatah, W. Strauss, Stability theory of solitary waves in the presence of symmetry, I, *J. Func. Anal.* **74** (1987), 160-197.

[92] P.N. Guzdar, B. Khubchandani, R. Roy, Influence of stochasticity on multiple four-wave-mixing processes in an optical fiber, *Phys. Rev. E* **66** (2002) 066609.

[93] A. Hasegawa, F. Tappert, Transmission of stationary nonlinear optical pulses in dispersive dielectric fibers, I; Anomalous dispersion, *Appl. Phys. Lett.* **23** (1973) 142-144.

[94] H.A. Haus, Y. Lai, Quantum theory of soliton squeezing: a linearized approach, *J. Opt. Soc. Am. B* **7** (1990) 386-392.

[95] D. Hochberg, C. Molina-Paris, J. Pérez-Mercader, M. Visser, Effective action for stochastic partial differential equations, *Phys. Rev. E* **60** (1999) 6343-6360.

[96] M.K. Heck, Principles of large deviations for the empirical processes of the Ornsetin-Uhlenbeck process, *J. Theoret. Probab.* **12** (1999) 61-75.

[97] K.P. Ho, Non-Gaussian statistics of soliton timing jitter induced by amplifier noise, *Opt. Lett.* **28** (2003) 2165–2167.

[98] F.X. Kärtner, D.J. Dougherty, H.A. Haus, E.P. Ipen, Raman noise and soliton squeezing, *J. Opt. Soc. Am. B* **11** (1994) 1267-1276.

[99] T. Kato, On Nonlinear Schrödinger Equation, *Ann. Inst. H. Poincaré Phys. Théor.* **46** (1987) 113-129.

[100] I.A. Khovanov, D.G. Luchinsky, R. Mannella, P.V.E. Mc Clintock, Fluctuational escape and related phenomena in nonlinear optical systems, *Modern Nonlinear Optics, Part 3* (Second Edition), John Wiley & Sons (2001) 469-524.

[101] B. Khubchandani, P.N. Guzdar, R. Roy, Influence of stochasticity on multiple four-wave-mixing processes in an optical fiber, *Phys. Rev. E* **66** (2002) 066609.

[102] L. Koralov, Random perturbations of 2-dimensional Hamiltonian flows, *Probab. Theory Related Fields* **129** (2004) 37-62.

[103] M. Ledoux, T. Lyons, Z. Qian, Lévy area of Wiener processes on Banach spaces, *Ann. Probab.* **30** (2002) 546-578.

[104] M. Ledoux, Z. Qian, T. Zhang, Large deviations and support theorem for diffusion process via rough paths, *Stochastic Process. Appl.* **102** (2002) 265-283.

[105] A. Lejay, M. Gubinelli, S. Tindel, Young integrals and SPDEs, preprint IECN, Nancy (2004) http://arxiv.org/abs/math.PR/0407294.

[106] D.G. Luchinsky, R.S. Maier, R. Mannella, P.V.E. McClintock, D.L. Stein, Experiments on critical phenomena in noisy exit problem, *Phys. Rev. Lett.* **79** (1997) 3109-3112.

[107] T.J. Lyons, Differential equations driven by rough signals, *Rev. Mat. Iberoamericana* **14** (1998) 215-310.

[108] T.J. Lyons, Z. Qian, *System Control and Rough Paths*. Oxford Univ. Press (2000).

[109] B. Maslowski, D. Nualart, Evolution driven by fractional Brownian motion, *J. Funct. Anal.* **202** (2003) 277-305.

[110] C.J. McKinstrie, T.I. Lakoba, Probability-density function for energy perturbation of isolated optical pulses, *Opt. Expr.* **11** (2003) 3628-3648.

[111] C.R. Menyuk, Non-Gaussian corrections to the Gordon-Haus distribution resulting from soliton interractions, *Opt. Lett.* **20** (1995) 285-287.

[112] A. Millet, M. Sanz-Solé, A simple proof of the support theorem for diffusion processes, *Lecture Notes in Math., Séminaire de Probabilités* **1583** (1994) 35-48.

[113] A. Millet, M. Sanz-Solé, Large deviations for rough paths of the fractional Brownian motion, published online in *Ann. Inst. H. Poincaré Probab. Statist.*

[114] R.O. Moore, G. Biondini, W.L. Kath, Importance sampling for noise-induced amplitude and timing jitter in soliton transmission systems, *Optics Letters* **28** (2003) 105-107.

[115] S. Munk, Large deviations w.r.t. quasi-every starting point for symmetric right processes on general state spaces, *Probab. Theory Related Fields* **99** (1994) 527-548.

[116] S. Peszat, Large deviation principle for stochastic evolution equations, *Probab. Theory Related Fields* **98** (1994) 113-136.

[117] S. Peszat, Exponential tail estimates for infinite-dimensional stochastic convolutions, *Bull. Pol. Acad. Sci.* **40** (1992) 323-333.

[118] J. Printems, Aspects théoriques et numériques de l'équation de Korteweg-de Vries stochastique, Thèse de l'Université Paris Sud (1998).

[119] P. Priouret, Remarques sur les petites perturbations de systèmes dynamiques, in *Séminaire de Probabilités XVI (1980/81)*. Springer-Verlag, Berlin, *Lecture Notes in Math.* **920** (1982) 184-200.

[120] A. Rouault, M. Yor, M. Zani, A LDP result for the strong Arc-Sine law, *J. Theor. Probability* **15** (2002) 793-815.

[121] C. Rovira, S. Tindel, Sharp Laplace asymptotics for a parabolic SPDE, *Stochastics Stochastics Rep.* **69** (2000) 11-30.

[122] C. Rovira, S. Tindel, Sharp large deviation estimates for a certain class of sets on the Wiener space, *Bull. Sci. Math.* **124** (2000) 525-555.

[123] C. Rovira, S. Tindel, Sharp large deviation estimates for the stochastic heat equation, *Potential Analysis* **14** (2001) 409-435.

[124] C. Rovira, S. Tindel, Sharp Laplace asymptotics for a hyperbolic SPDE, in *Stochastic analysis and related topics* **VII**, edited by L. Decreusefond, B. Oskendall, A.S. Üstünel, Birkhauser, Boston (2001) 225-244.

[125] R. Sàsik, L.M.A. Bettencourt, S. Habib, Thermal vortex dynamics in two-dimensional condensate, *Phys. Rev. B* **62** 1238-1243.

[126] M. Sanz-Solé, Mathematical models for stochastic evolutions, *Quartes Jornades d'investigació i foment de la Multidisciplinarietat*, Universitat Politècnica de València (2002) 1-29.

[127] J. Shatah,W. Strauss, Instability of nonlinear bound states, *Commun. Math. Phys.* **100** (1985) 173-190.

[128] V.N. Smelyanskiy, M.I. Dykman, Optimal control of large fluctuations, *Phys. Rev. E* **55** (1997) 2516-2521.

[129] V.N. Smelyanskiy, M.I. Dykman, H. Rabitz, B.E. Vugmeister, Fluctuations, escape, and nucleation in driven system: logarithmic susceptibility, *Phys. Rev. Lett.* **79** (1997) 3113-3116.

[130] A. Soffer, M.I. Weinstein, Selection of the ground state for nonlinear Schrödinger equations, to appear in *Rev. Math. Phys.*

[131] R.B. Sowers, Large deviations for a reaction-diffusion equation with non-Gaussian perturbations, *Ann. Probab.* **20** (1992) 504–537.

[132] R. B. Sowers, Large deviations for the invariant measure of a reaction-diffusion equation with non-Gaussian perturbations. *Probab. Theory Related Fields* **92**, (1992) 393-421.

[133] R.B. Sowers, Stochastic averaging with a flattened Hamiltonian: a Markov process on a stratified space (a whiskered sphere), *Trans. Amer. Math. Soc.* **354** (2001) 853-900.

[134] R.B. Sowers, Stochastic averaging near homoclinic orbits with multiplicative noise, *Stoch. Dyn.* **3** (2003), 299-391.

[135] R.B. Sowers, A boundary layer theory for diffusively perturbed transport aroud a heteroclinic cycle, *Comm. on Pure and Appl. Math.* **152** (2004) 299-391.

[136] C. Sulem, P.L. Sulem, *The Nonlinear Schrödinger Equation, Self-Focusing and Wave Collapse*. Springer-Verlag, New York, *Appl. Math. Sciences* (1999).

[137] S. Tindel, C.A. Tudor, F. Viens, Stochastic evolution equations with fractional Brownian motion, *Probab. Theory Related Fields* **127** (2003) 186-204.

[138] M. Tsutsumi, Nonexistence of global solutions to the Cauchy problem for the damped nonlinear Schrödinger equations, *SIAM J. Math. Anal.* **15** (1984) 357-366.

[139] J.B. Walsh, An introduction to stochastic partial differential equations, in *École d'été de Probabilité de Saint-Flour*, edited by P.L. Hennequin. Springer-Verlag, Berlin, *Lecture Notes in Math.* **1180** (1986) 265-439.

[140] M.I. Weinstein, Lyapounov stability of ground sates of nonlinear dispersive evolution equations,*Comm. on Pure and Appl. Math.* **39** (1986) 51-68.

[141] A.N. Yannacopoulos, D.J. Frantzeskakis, C. Polymis, K. Hizanidis, Motion of 2D Schrödinger solitary waves in the presence of random external potentials, *Physica Scripta* **65** (2002) 363–368.

Grandes déviations pour des équations de Schrödinger non linéaires stochastiques et applications

Résumé: Dans cette thèse nous étudions l'asymptotique de petits bruits pour des perturbations aléatoires d'équations de Schrödinger non linéaires. Les bruits sont Gaussiens, la plupart du temps blancs en temps et toujours colorés en espace, additifs ou multiplicatifs. Un évènement de grandes déviations est un évènement où le système diffère significativement du système déterministe. Lorsque le bruit tend vers zéro, la probabilité d'un tel évènement rare tend vers zéro sur une échelle logarithmique avec pour vitesse l'amplitude du bruit. Nous prouvons des principes de grandes déviations trajectoriels. Dans ce cas le facteur multiplicatif de la vitesse, le taux, est relié à un problème de contrôle optimal. Les résultats sont appliqués aux temps d'explosion. Nous étudions ensuite l'asymptotique de petits bruits des queues de la masse et de la position du signal dans une "limite bruit blanc". Les fluctuations de ces quantités sont les causes principales d'erreur de transmission par solitons dans les fibres optiques. Nous considérons également le problème des temps moyens et des points de sortie d'un voisinage de zéro pour des équations faiblement amorties. Enfin, nous présentons un principe de grandes déviations et un théorème de support pour des bruits Gaussiens fractionnaires additifs.

Mots clés: Equation de Schrödinger non linéaire, équations aux dérivées partielles stochastiques, grandes déviations, ondes solitaires, explosion en temps fini, mouvement Brownien fractionaire, théorèmes de support.

Large deviations for stochastic nonlinear Schrödinger equations and applications

Abstract: This thesis is dedicated to the study of the small noise asymptotic in random perturbations of nonlinear Schrödinger equations. The noises are Gaussian, mostly white in time and always colored in space, of additive and multiplicative types. Large deviations are such that the behavior of the stochastic system differs significantly from the deterministic one. As the noise goes to zero the probability of such rare events goes to zero on a logarithmic scale with speed given by the noise amplitude. We prove large deviation principles at the level of paths. The rate of convergence to zero of the logarithm of the probabilities is related to an optimal control problem. Our first application is to the blow-up times. We then apply our results to the study of the small noise asymptotic of the tails of the mass and position of the soliton-like pulse in a "white noise limit". The fluctuations of these quantities are the main causes of error in optical soliton transmission. We also consider the problem of the mean exit times and the exit points from a neighborhood of zero for weakly damped equations. Finally we present large deviations and a support theorem for fractional additive Gaussian noises.

Keywords: Nonlinear Schrödinger equation, large deviations, stochastic partial differential equations, solitary waves, blow-up, fractional Brownian motion, support theorems.

2000 Mathematics Subject Classification: 35Q55, 35Q51,60H15, 60F10, 78A60.